ESTIMATION OF MEANS FOR INVERSE GAUSSIAN IN THE PRESENCE OF UNCERTAIN PRIOR INFORMATION

by

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ABSTRACT

ESTIMATION OF MEANS FOR INVERSE GAUSSIAN IN THE PRESENCE OF UNCERTAIN PRIOR INFORMATION

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The University of Houston Clear Lake, 1996

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An investigator may want estimates of the means of a multivariate distribution under the hypothesis that they are all the same. If the assumption of the null hypothesis is not true, there may be a higher than acceptable risk in deciding between the restricted maximum likelihood estimator (RMLE) that is associated with the null hypothesis and the unrestricted maximum likelihood estimator (UMLE) that is associated with the alternate hypothesis. To decrease this risk, it is prudent to use either a preliminary test estimator (PTE) or a shrinkage estimator (SE) when obtaining parameter estimates based on uncertain *a priori* information. Unlike the RMLE, both of these estimators are more robust to departures from the null hypothesis. This thesis develops a preliminary test estimator (PTE) and a shrinkage estimator (SE) for the inverse Gaussian distribution and evaluates their risks, relative efficiencies and power function.

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INTRODUCTION

Purpose of the Investigation

In many situations, such as in the study of product reliability, the distribution of the random variable (for example, failure time) is likely to be skewed. One distribution that may be appropriate in many of these cases is the Inverse Gaussian. In the first place, it possesses sampling distribution theory similar to that of the normal distribution. Secondly, it represents a wide range of distributions with respect to skewness. This provides the investigator with easy-to-use statistical methods that are not available with other more commonly used skewed distribution functions such as the Weibull. These include one- and two-sample *t* tests, analysis of variance, confidence intervals, and regression analysis as described in the book by Chhikara and Folks (1989).

The objective of this investigation is to evaluate estimators for Inverse Gaussian parameters in the presence of uncertainty in the equality of their values and to evaluate their risks, relative efficiencies and power function.

Statement of the Problem

When comparing different populations, distributions or mechanisms, it is sometimes necessary to obtain parameter estimates that have prior knowledge incorporated. This is particularly the case when one suspects, with some amount of uncertainty, that the means have the same value.

Ali and Saleh (1989, 1991) have addressed the problem of estimating the mean parameters for the multivariate normal population case. They were "primarily concerned with the estimation of the mean vector μ, when it is suspected but not sure that $\mu_1 = \mu_2 = \ldots = \mu_p = \mu$ (unknown), i.e., with uncertain prior information about μ." They developed bias and risk expressions for a preliminary test estimator (PTE) and a shrinkage estimator (SE) and evaluated their performance.

There currently does not exist a similar development and analysis for the Inverse Gaussian case. The purpose of this thesis is to develop general expressions for risks and relative efficiencies for a preliminary test estimator and a shrinkage estimator for the Inverse Gaussian case and to compare them. In addition, this thesis develops and evaluates the power function for a wide range of Inverse Gaussian distributions.

Details of the Investigation

The Inverse Gaussian distribution function represents a family of populations that range from the highly skewed to those that are almost normal. The probability distribution function for an Inverse Gaussian distributed random variable X is

$$f(x;\mu,\lambda) = \sqrt{\frac{\lambda}{2\pi}}x^{-3/2}\exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right), x > 0$$

where μ>0 is the mean of the distribution and λ>0 is a scale parameter.

For this discussion, let X_{ij}, $j = 1,2,\ldots,n_i$, be independent Inverse Gaussian random variables with mean θ_i, and common scale parameter λ, $i = 1,2,\ldots p$. Let the mean vector of these p distributions be denoted by $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_p)'$.

For the one-way classification process, this thesis considers only two possible types of joint distributions for X_{ij}:

(1) One for which there is uncertain *a priori* knowledge that the means are the same with common λ, and

(2) One for which the means are different and unspecified, with common λ.

Let $H_0 : \boldsymbol{\theta} = \theta \mathbf{1}_p' = \theta(1, \;\; 1, \;\; \ldots, \;\; 1)' = (\theta, \;\; \theta, \;\; \ldots, \;\; \theta)'$ represent the null hypothesis associated with the first case, and let $H_1 : \boldsymbol{\theta} \neq \theta \mathbf{1}_p'$ represent the alternate (non-null) hypothesis associated with the second case.

The one-way classification model for the data is

$$X_{ij} = \theta + \delta_i + \xi_{ij}; \quad j = 1,2,\ldots n_i \quad , \quad i = 1,2,\ldots,p$$

where $\delta_i = (\theta_i - \theta)$ is the difference between the i^{th} mean and the overall mean. A zero value for δ_i implies the distribution is for the null hypothesis. On the other hand, a non-zero value for δ_i implies the distribution is a non-null distribution; that is, a distribution for which the null hypothesis is false. Also, the random variable ξ_{ij}, $j = 1,2,\ldots,n_i$, $i = 1,2,\ldots,p$, is distributed as Inverse Gaussian with mean θ_i and scale parameter λ. That is,

$$\xi_{ij} \approx IG(\theta_i, \lambda).$$

THEORETICAL BACKGROUND

Maximum Likelihood Estimators

The first case is the restricted case of the null hypothesis, H_o (the equality of means). The likelihood function under H_o given the data $x_i = (x_{i1}, x_{i2}, \ldots, x_{in_i})'$ for $i = 1,2,\ldots,p$, is

$$L(x_i|\theta) = \prod_{i=1}^{p}\prod_{j=1}^{n_i}\left[\sqrt{\frac{\lambda}{2\pi}}x_{ij}^{-3/2}\exp\left(-\frac{\lambda(x_{ij}-\theta)^2}{2\theta^2 x_{ij}}\right)\right].$$

The restricted maximum likelihood estimators (RMLE) of θ and λ are given by Chhikara and Folks (1989) as

$$\hat{\theta} = \bar{X}$$

$$\frac{1}{\hat{\lambda}} = \frac{1}{n}\sum_i\sum_j\left(\frac{1}{X_{ij}} - \frac{1}{\bar{X}}\right)$$

where $n = \sum_i n_i$, $\bar{X}_i = \sum_j X_{ij} / n_i \Rightarrow \bar{X} = \sum_i n_i \bar{X}_i / n$.

So the maximum of the likelihood for the restricted situation of equal means is

$$L_0(\hat{\theta}, \hat{\lambda}) = \left(\frac{1}{2\pi e}\right)^{n/2}\prod_i\prod_j x_{ij}^{-3/2}\left(\hat{\lambda}\right)^{-n/2}$$

For the alternate hypothesis, the likelihood function for the parameters is

$$L(x_i|\theta) = \prod_{j=1}^{n}\left[\sqrt{\frac{\lambda}{2\pi}}x_{ij}^{-3/2}\exp\left(-\frac{\lambda(x_{ij}-\theta_i)^2}{2\theta_i^2 x_{ij}}\right)\right], \quad i = 1,2,\ldots,p$$

The unrestricted maximum likelihood estimators (UMLE) of $\theta = (\theta_1, \theta_2, \ldots \theta_p)'$

and λ are given by

$$\tilde{\theta}_i = \overline{X}_i, \quad i = 1,2,\ldots p$$

$$\frac{1}{\tilde{\lambda}} = \frac{1}{n}\sum_i\sum_j\left(\frac{1}{X_{ij}} - \frac{1}{\overline{X}_i}\right)$$

So the maximum of the likelihood for the general case is

$$L_1(\tilde{\theta}_i, \tilde{\lambda}) = \left(\frac{1}{2\pi e}\right)^{n/2}\prod_i\prod_j x_{ij}^{-3/2}\left(\tilde{\lambda}\right)^{-n/2}$$

Hence, the likelihood ratio $L = L_0 / L_1$ leads to $L^{2/n} = \dfrac{\sum_i\sum_j\left(\frac{1}{x_{ij}} - \frac{1}{\bar{x}_i}\right)}{\sum_i\sum_j\left(\frac{1}{x_{ij}} - \frac{1}{\bar{x}}\right)}$

The denominator can be decomposed into the sum of two independently distributed χ^2 variates (Chhikara and Folks - 1989), that is

$$\sum_i\sum_j\left(\frac{1}{x_{ij}} - \frac{1}{\bar{x}}\right) = \sum_i\sum_j\left(\frac{1}{x_{ij}} - \frac{1}{\bar{x}_i}\right) + \sum_i n_i\left(\frac{1}{\bar{x}_i} - \frac{1}{\bar{x}}\right)$$

where

$$\sum_i\sum_j\left(\frac{1}{x_{ij}} - \frac{1}{\bar{x}_i}\right) \sim \left(\frac{1}{\lambda}\right)\chi^2_{n-p}$$

and

$$\sum_i n_i\left(\frac{1}{\bar{x}_i}-\frac{1}{\bar{x}}\right) \sim \left(\frac{1}{\lambda}\right)\chi^2_{p-1}$$

One of these variates is the same as the component in the numerator.

Therefore, the likelihood ratio test statistic for testing the equality of p Inverse Gaussian means is

$$W = \frac{\sum_i n_i\left(\frac{1}{\bar{x}_i}-\frac{1}{\bar{x}}\right)/(p-1)}{\sum_i\sum_j\left(\frac{1}{x_{ij}}-\frac{1}{\bar{x}_i}\right)/(n-p)} \sim F_{p-1,n-p}$$

It follows that the α level rejection region is given by $W > F_{p-1,n-p,\alpha}$ where $F_{p-1,n-p,\alpha}$ is the 100(1 - α) percentage point of the *F* distribution with (p - 1) and (*n* - *p*) degrees of freedom.

__Pre-Test Estimators__

When the null hypothesis (H_o) holds, the estimate $\hat{\theta}$ for the restricted case will have a smaller risk under quadratic loss than the estimate $\tilde{\theta}$ for the unrestricted case. On the other hand, when H_o may not hold and *W* is near the critical point, there may be greater risk in making the correct decision than is acceptable. As a result, when the prior information on H_o is rather uncertain, it is desirable to use either a preliminary test estimator (PTE) or a shrinkage estimator (SE) since they are more robust to departures from the null hypothesis and have bounded risks.

The shrinkage, or "Stein Rule Estimator" (or just SE), is defined as follows:

$$\hat{\theta}_n^S = \hat{\theta}_n + [1 - c/W]\left(\tilde{\theta}_n - \hat{\theta}_n\right).$$

where c is the shrinkage constant and $\hat{\theta}_n = \left(\bar{X},\ \ \bar{X},\ \ \ldots,\ \ \bar{X}\right)'$ and $\tilde{\theta}_n = \left(\bar{X}_1,\ \ \bar{X}_2,\ \ \ldots,\ \ \bar{X}_p\right)'$. For the numerical analysis in the investigation, c was arbitrarily selected as the same value used by Ali and Saleh (1991). That is, $c = (p - 3)/(m + 2)$, where $m = n - p$.

The preliminary test estimator, or PTE, is defined as follows:

$$\hat{\theta}_{n(\alpha)}^P = \hat{\theta}_n + \left[1 - I\left(W \le F_{p-1,n-p,\alpha}\right)\right]\left(\tilde{\theta}_n - \hat{\theta}_n\right);$$

where $I(A)$ is the indicator function for the set A. That is, as the null hypothesis is accepted using a specified α level of significance, $I(W \le F_{p-1,n-p,\alpha}) = 1$. In this case, the preliminary test estimate selected is the RMLE. Otherwise, if the null hypothesis is not accepted the preliminary test estimate is the UMLE.

The SE transitions smoothly across the critical α point, whereas, the PTE exhibits a step change.

Ali and Saleh (1991) evaluated maximum efficiencies of these estimators for the multivariate normal distribution case over a broad range of critical regions. To facilitate this they used analytic formulas for the bias and risks associated with each of the estimators.

This investigation focuses on comparing these estimators for the Inverse Gaussian means considering their risks under quadratic loss, relative efficiencies and power function. It includes an analysis of one-way classification using Monte Carlo simulations of a wide range of Inverse Gaussian distributions and provides conclusions that draw on the theoretical development and on their performance with the simulated data.

The Evaluation Approach

The formulas used by Ali and Saleh are a direct consequence of two facts. First, linear combinations of *k* normally distributed random variables are normally distributed. Second, the test statistics for the non-null distribution for the normal case are non-central χ^2 with *k degrees of freedom* and noncentrality parameter τ^2, where *k* is the number of means to be considered in a point estimate.

For the density function for a non-central χ^2 random variable *V*, one may refer to Judge & Bock (1978).

The method used by Ali and Saleh can be briefly described as follows: Let *W* be a *k*-dimensional normally distributed random variable with mean vector δ and variance-covariance matrix I_k. Here, δ is a *k x 1* difference vector between the unrestricted mean and the restricted mean of the null hypothesis and I_k is a *(k x k)* identity matrix. If the test statistic is $V = W^2$, then the expected value used in the pre-test estimator bias computation is

$$E\left[I_{(0,c)}(v)w\right] = \delta E\left[I_{(0,c)}(\chi^2_{k+2,\tau^2})\right] = \delta \Pr\left[\chi^2_{k+2,\tau^2} < c\right],$$

where *c* is the critical value of the test, $I_{(0,c)}(\cdot)$ is an indicator function for $(\cdot)$ and $\tau^2 = \delta'\delta/2$ is the noncentrality parameter associated with *V*. This result follows since any non-central chi-square distributed random variable χ^2_{k,τ^2} with *k* *degrees of freedom* and noncentrality parameter τ^2 can be expressed as a central chi-square random variable $\chi^2_{k+2\beta,0}$ with *(k+2β) degrees of freedom* (conditioned on *β*), where *β* is a Poisson distributed random variable with parameter τ^2 (Judge & Bock (1978), Theorem A.2.21, Appendix A).

This result and a similar theorem for mean square error (MSE) of *W* allowed Ali and Saleh to evaluate, in closed-form, the bias and risks associated with pre-test estimators for the multivariate normal population case.

For the Inverse Gaussian case, linear combinations of Inverse Gaussian random variables are not Inverse Gaussian without certain restrictions on the relationship between the parameters. Furthermore, the test statistic for a univariate non-null distribution is distributed as a *nonlinear weighted* non-central χ^2 random variable with 1 *degree of freedom* (Chhikara and Folks). The distribution of the test statistic for the multivariate case is unknown.

Hence, there is no analogous theory to provide closed-formulas for the Inverse Gaussian distribution. So, one must resort to alternative methods to evaluate risk associated with the estimators for non-null conditions. The method selected for this study is to generate representative distributions using Monte Carlo simulations and then evaluate the functions numerically.

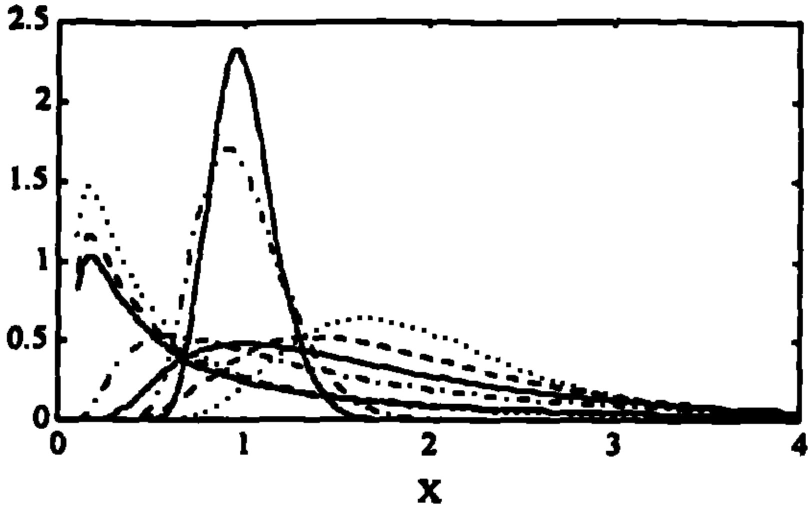
Table 1 presents the set of distributions considered for this study. These distributions span values of $\varphi = \lambda/\theta$ from 0.125 (highly skewed) to 32.0 (nearly normal). The *Mathematica* computer program was use to produce all of the data for this thesis.

Also included in Table 1 is the size of the data set for each Monte Carlo simulation. To get the same level of accuracy in the estimates the highly skewed distributions require much larger data sets than the nearly normal ones. Figure 1 illustrates the Inverse Gaussian density functions for the distributions defined in Table 1.

Table 1. Input for Monte Carlo Simulations

φ	λ	θ	δ	Data Size		φ	λ	θ	δ	Data Size
0.125	0.5	4.0	0.0	16050		4.0	8	2.0	0.0	1750
			0.5	27650					0.5	1400
			1.0	27000					1.0	1400
0.25	0.5	2.0	0.0	3500		8.0	16	2.0	0.0	1400
			0.5	3850					0.5	1400
			1.0	6480					1.0	1400
0.5	0.5	1.0	0.0	3850		16.0	16	1.0	0.0	1400
			0.5	1750					0.5	1400
			1.0	3500					1.0	1400
1.0	2	2.0	0.0	3500		32.0	32	1.0	0.0	1400
			0.5	1750					0.5	1400
			1.0	2100					1.0	1750
2.0	4	2.0	0.0	1400						
			0.5	1400						
			1.0	1400						

Figure 1. Inverse Gaussian Density Functions for the Values of (φ, θ) in Table 1.



Note: $X \approx IG(\theta, \varphi\theta)$ for each distribution.

The plots in Figure 1 correspond to the values of φ as follows:

φ	Plot Symbol
0.125	——
0.25	– –
0.50	. . .
1.0	–·
2.0	——
4.0	– –
8.0	. . .
16.0	–·
32.0	——

Risk and Relative Efficiency Estimation

The risk of an estimator $\hat{\theta}_n$ associated with quadratic loss is defined as the expected value of the quadratic loss function for the estimator, i.e.,

$$R\left(\hat{\theta}_n,\theta\right)=E\left[L\left(\hat{\theta}_n,\theta\right)\right],$$

where the quadratic loss function is defined as follows:

For a j^{th} random sample, the *quadratic loss function* of an estimator $\hat{\theta}_n$ is

$$L_j\left(\hat{\theta}_n,\theta\right)=\sum_{i=1}^{p}w_i\left(\hat{\theta}_{n_i}-\theta_i\right)_j^2,\qquad j=1,2,\ldots,m,\qquad \text{where } w_i=n_i/\sigma_i^2\,.$$

For a multivariate normal distribution, this function is distributed as central χ^2 random variable when the null hypothesis of equal means is true, and non-central χ^2 otherwise. However, for a multivariate Inverse Gaussian distribution, this property does not hold. So, a numerical estimate based on *m* random samples must be obtained.

An estimate of this risk is the average of the loss function over *m* random samples:

$$\tilde{R}\left(\hat{\theta}_n,\theta\right)=\frac{1}{m}\sum_{j=1}^{m}L_j\left(\hat{\theta}_n,\theta\right)=\frac{1}{m}\sum_{j=1}^{m}\sum_{i=1}^{p}w_i\left(\hat{\theta}_{ni}-\theta_i\right)_j^2$$

where $w_i=n_i/\sigma_i^2=n_i/\left(\theta_i^3/\lambda\right)=n_i\lambda/\theta_i^3$ for the Inverse Gaussian case.

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where $w_i=n_i/\sigma_i^2=n_i/\left(\theta_i^3/\lambda\right)=n_i\lambda/\theta_i^3$ for the Inverse Gaussian case.

The relative efficiencies of the estimators with respect to the unrestricted maximum likelihood estimator (UMLE) are expressed as the ratios of the risk of the UMLE to that of the RMLE, SE and PTE, respectively. That is,

$$E_{RMLE} = R(\tilde{\theta}_n, \theta) \big/ R(\hat{\theta}_n, \theta)$$

$$E_{SE} = R(\tilde{\theta}_n, \theta) \big/ R(\hat{\theta}_n^S, \theta)$$

$$E_{PTE(\alpha)} = R(\tilde{\theta}_n, \theta) \big/ R(\hat{\theta}_{n(\alpha)}^P, \theta)$$

are the relative efficiencies of the RMLE, SE and PTE (using a specified α significance level), respectively.

Power Function Estimation

The power function $K(\theta)$ of an estimator $\hat{\theta}$ is the probability of rejecting the null hypothesis H_0 in favor of the distribution under consideration. One way of expressing this mathematically is

$$K(\theta) = \Pr\left[W > F_{p-1,n-p,\alpha} | \hat{\theta}\right].$$

Let *N* be the size of the simulated population being considered such that $H_0: \hat{\theta} = \theta 1'_p$ and $H_1: \hat{\theta} \neq \theta 1'_p$, where p is the number of parameters to be

ANALYSIS OF RESULTS

Approach and Data Organization

Since the parameter $\varphi = \lambda/\theta$ is frequently used to indicate the skewness of an Inverse Gaussian distribution, the data generated for this thesis are organized to cover a spectrum from highly skewed (small φ) to almost normal (large φ) distributions. Nine values of φ were arbitrarily selected to cover the spectrum from 0.125 to 32.0. They are 0.125, 0.25, 0.5, 1.0, 2.0, 4.0, 8.0, 16.0 and 32.0. For each value of φ there are four values of p (4, 5, 6, 7). For each value of p there are three values of δ (0.0, 0.5, 1.0). Data for δ = 0.0 represents results for a null distribution; whereas, the data for δ = 0.5 and 1.0 represent results for non-null distributions. For each distribution simulated, only one of the p means has the value $\theta_i = \theta + \delta_i$. All other p - 1 means are equal to θ.

Figures 2 through 9 summarize the results obtained in this analysis. The Monte Carlo simulated distributions summarized in Table 1 provided the inputs to these analyses. Each figure presents plots of selected parameters over the spectrum of φ. Figures 2 and 3 summarize the risk of the UMLE for the three values of δ for p = 4 and 6, respectively. Figures 4 and 5 illustrate the relative efficiencies of the RMLE, SE and PTE (for $\alpha = 0.05$) for the non-null distribution for $\delta = 0.5$ for p = 4 and 6, respectively. Figures 6 and 7 present similar results for the non-null distribution for $\delta = 1.0$. Finally, Figures 8 and 9 present examples of the power function (for $\alpha = 0.05$) for the non-null distributions for n = 25 and for p = 4 and 6, respectively.

Evaluation and Interpretation

A noteworthy result of the data for the non-null distributions for $\varphi < 0.5$ is that the rejection rate is the same as for the null distributions. This is clearly shown by the power function data in Figures 8 and 9. Furthermore, the relative efficiencies of the RMLE shown in Figures 4 through 7 are generally greater than any of the other estimators. This is consistent with acceptance of the null hypothesis.

Of course, this is not surprising for small φ's because of the large variability in the data for those distributions. To achieve the accuracy that is required to distinguish differences in means requires large sample sizes (refer to Table 1).

For example, to distinguish a $\delta = 1.0$ difference in means for a distribution with $\varphi = 0.125$ and $\theta = 4.0$ requires at least 128 samples. That is,

$$n \geq \sigma^2/\delta^2 = \theta^2/\left(\delta^2\phi\right) = 16/\left(1.0^2 \times 0.125\right) = 128.$$

Likewise, to distinguish a 0.5 difference requires over 512 samples.

For φ values between 0.5 and 2.0, rejections of the null hypothesis for non-null distributions are mixed, particularly for $\delta = 0.5$. Although the rejection rate is too small on the low end of the φ spectrum, it generally improves as φ increases, particularly as n also increases. For δ = 1.0, the results are more distinct.

In most cases for this range of φ, the RMLE for δ = 0.5 has less risk than the other estimators. However, as φ increases the PTE sometimes is equally efficient to the RMLE. This is shown in Figures 4 through 7.

On the other hand, the picture for the δ = 1.0 case changes earlier in this φ range. In either case, the efficiency of the RMLE drops off rapidly.

Furthermore, the data shows the SE beginning to dominate the RMLE and the PTE in terms of efficiency as φ increases; that is, as the Inverse Gaussian becomes more normal. The situation continues to progress in this manner. In fact, for φ = 4.0 and δ = 1.0, the dominance of the SE is almost total.

The values of the risks shown in Figure 2 are the composite UMLE risks, or the weighted average of the risks over the 7 sample sizes considered. That is, for each $n_k = (9,12,15,25,30,40,50)$ with p = 4 there is an estimate of the UMLE risk, say $\tilde{R}\left(\tilde{\theta}_n,\theta\right)_k$, $k = 1,2,\ldots,7$. The weighted average of the risk is determined by the following formula:

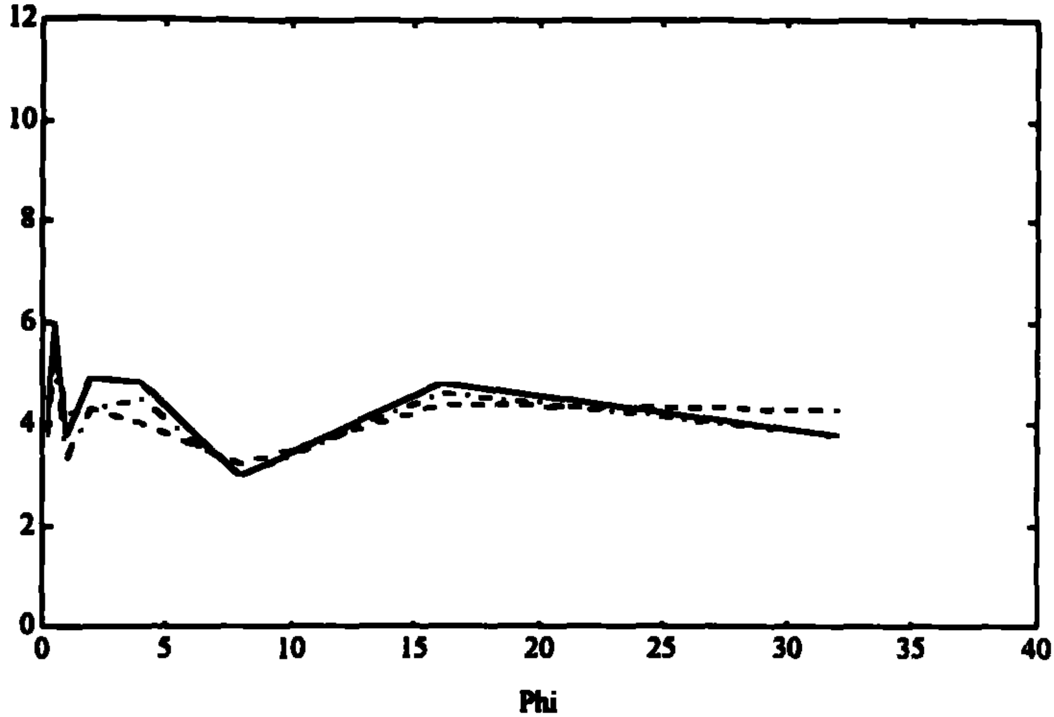
$$\tilde{R}\left(\tilde{\theta}_n,\theta\right) = \frac{1}{n}\sum_{k=1}^{7} n_k \tilde{R}\left(\tilde{\theta}_n,\theta\right)_k, \quad n = \sum_{k=1}^{7} n_k = 181$$

The "dip" in the UMLE risk profiles in Figures 2 and 3 also requires an explanation. The choice of the distributions corresponding to the φ-values of 4, 8, and 16 yield variances of 1, 1/2, and 1/16, respectively. However, the Monte Carlo random selection process produced a smaller mean-square-error (MSE) the mean estimators for the simulated distribution corresponding to $\varphi = 8$ than for the other two. One would not expect this to always occur. In this case it is an artifact of the random selection process.

Therefore, the resulting smaller risk (average MSE) at the $\varphi = 8$ point produces a dip in the profiles. The risks for the non-null cases exhibit the same trend as for the null distribution case, since 3 means in the group of 4 case (Figure 2) and 5 in the group of 6 case (Figure 3) have the same variability.

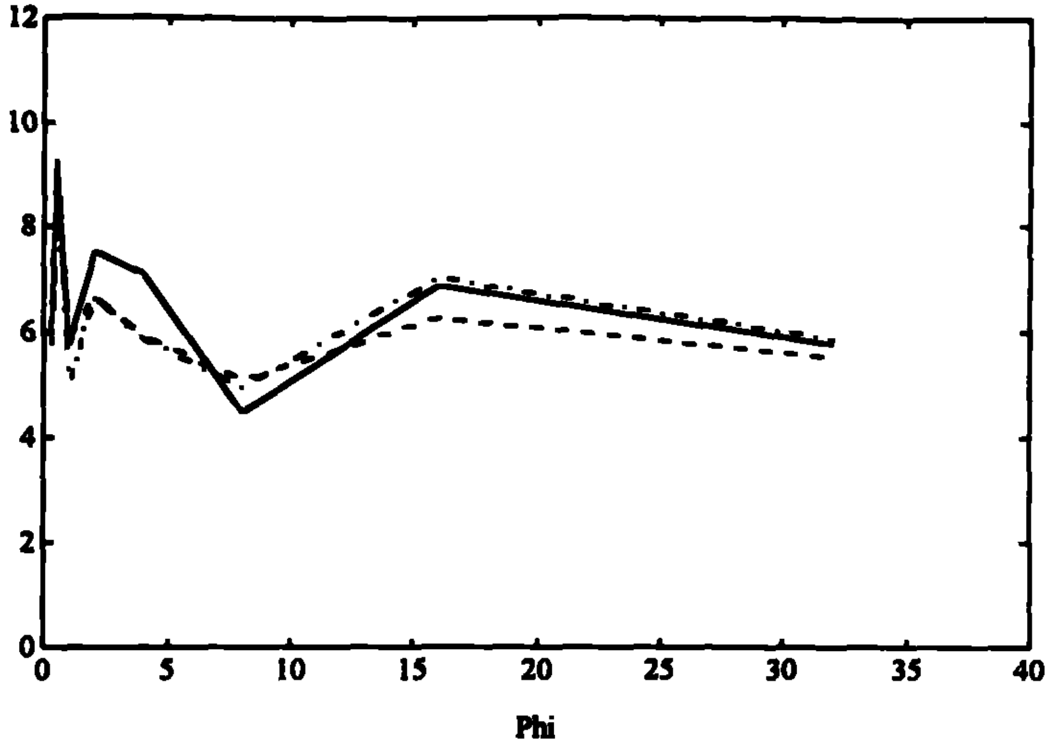
Given that there are no closed-form theoretical expressions for risk and efficiencies for the multivariate Inverse Gaussian case, the most significant result of the investigation is the agreement of the results to those for the normal case. In general, the risk of the UMLE is fairly equal to the number of Inverse Gaussian distributions considered as illustrated in Figures 2 and 3. This result also holds for the multivariate normal populations as shown theoretically by Ali and Saleh. Furthermore, the behavior of the relative efficiency data for the RMLE, SE and PTE as δ increases is also consistent to that for the normal case. Overall, the risk and efficiency data for the wide range of Inverse Gaussian distributions considered confirms the conclusions drawn by Ali and Saleh for the normal distribution case.

Figure 2. UMLE Risk Versus φ for p = 4



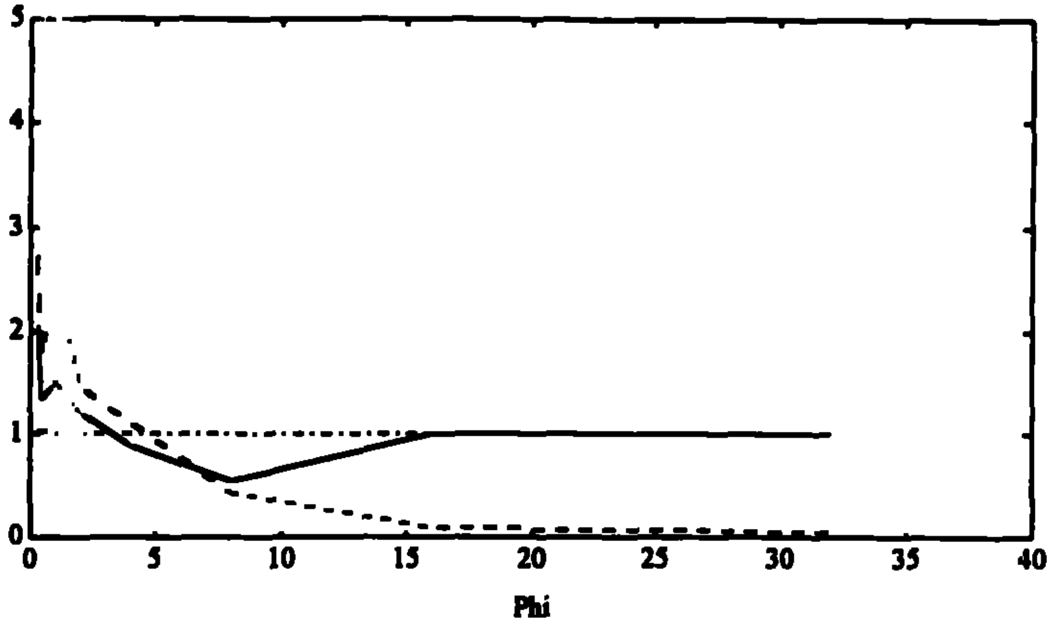
Note: __ is for δ = 0.0, -. is for δ = 0.5, and - - is for δ = 1.0.

Figure 3. UMLE Risk Versus φ for p = 6



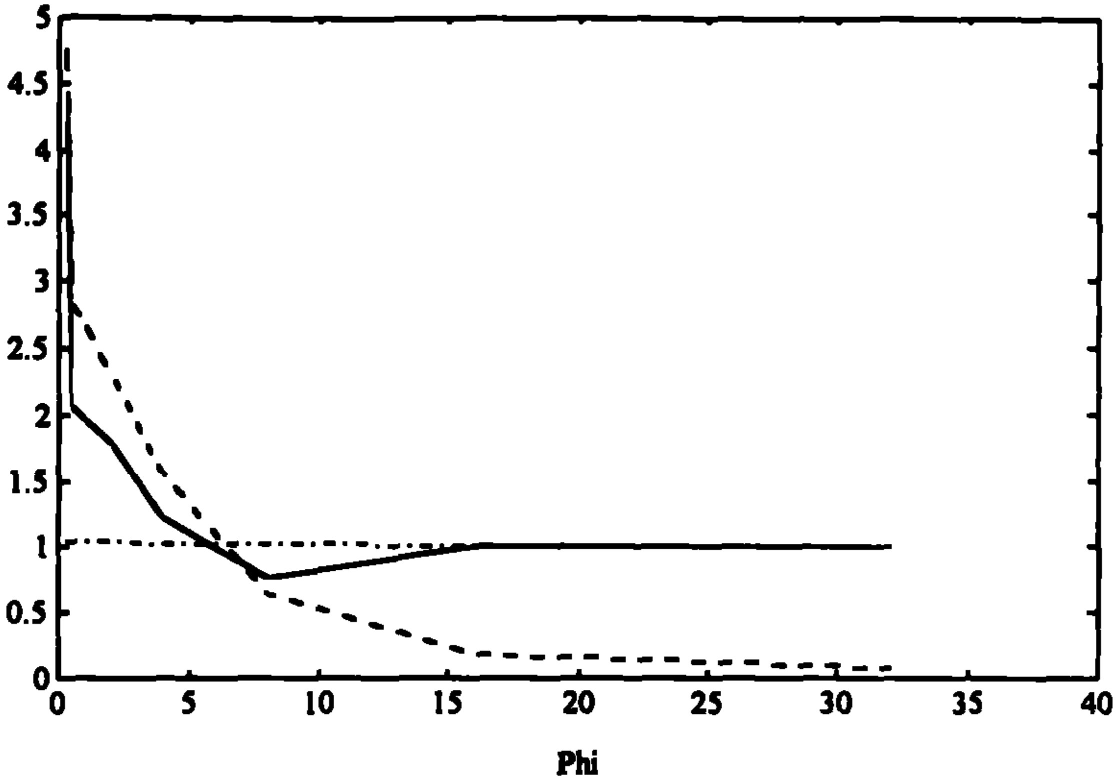
Note: __ is for δ = 0.0, -. is for δ = 0.5, and - - is for δ = 1.0.

Figure 4. Relative Efficiencies of RMLE, SE and PTE Versus φ for p = 4, n = 25, and δ = 0.5



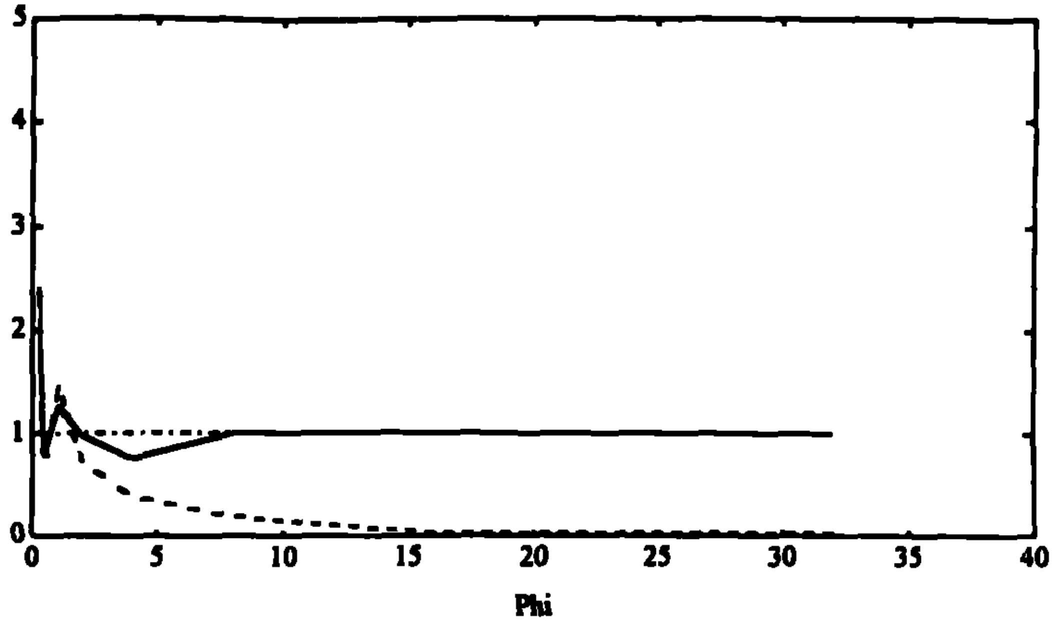
Note: __ is for PTE, -. is for SE, and - - is for RMLE.

Figure 5. Relative Efficiencies of RMLE, SE and PTE Versus φ
for p = 6, n = 25, and δ = 0.5



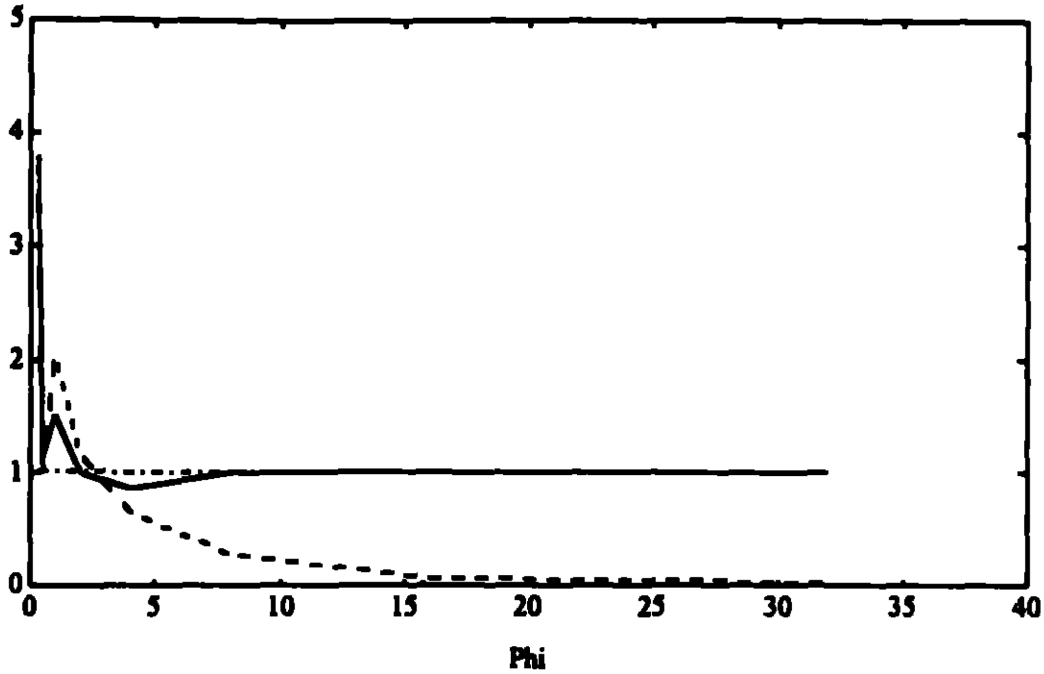
Note: __ is for PTE, -. is for SE, and - - is for RMLE.

Figure 6. Relative Efficiencies of RMLE, SE and PTE Versus φ for p = 4, n = 25, and δ = 1.0



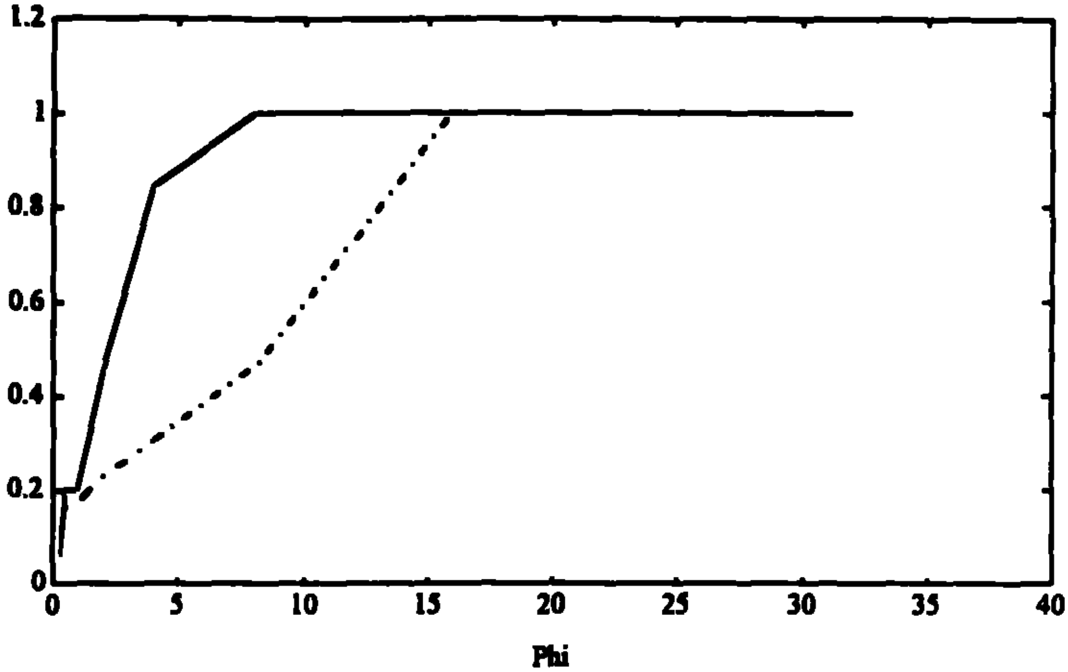
Note: __ is for PTE, -. is for SE, and - - is for RMLE.

Figure 7. Relative Efficiencies of RMLE, SE and PTE Versus φ for p = 6, n = 25, and δ = 1.0



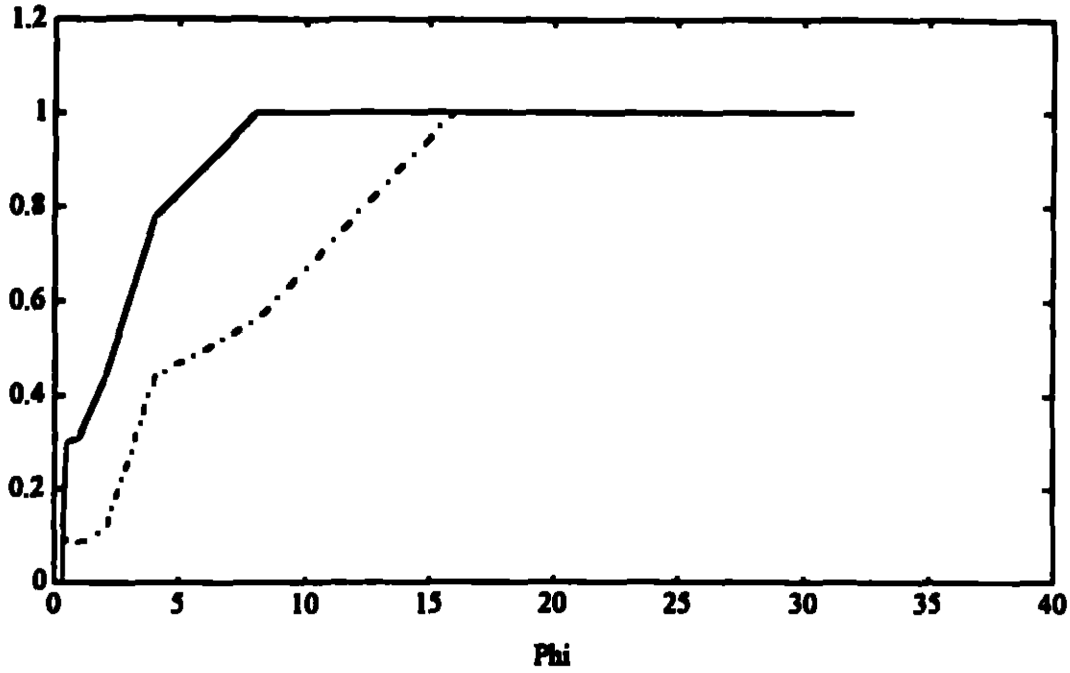
Note: __ is for PTE, -. is for SE, and - - is for RMLE.

Figure 8. Power Function at 0.05 Significance Level Versus φ for p = 4, n = 25, and δ = 0.5 and 1.0



Note: – is for δ = 0.5, and __ is for δ = 1.0.

Figure 9. Power Function at 0.05 Significance Level Versus φ for p = 6, n = 25, and δ = 0.5 and 1.0



Note: -- is for δ = 0.5, and __ is for δ = 1.0.

SUMMARY AND CONCLUSIONS

The primary objective of this thesis was to investigate the relative behavior of various estimators of the means from Inverse Gaussian distributions in the presence of uncertainty in their suspected values. To accomplish this objective, this study compared risks and relative efficiencies for four estimators over a wide range of Inverse Gaussian distributions. These estimators are a restricted maximum likelihood estimator (RMLE), the unrestricted maximum likelihood estimator (UMLE), a pre-test estimator (PTE), and a shrinkage estimator (SE). Since there currently is no theory to provide analytical formulas for estimator risks, relative efficiencies and power function for the Inverse Gaussian distribution, this thesis used numerical (instead of purely theoretical) methods to compare the behavior of the estimators. The specific method selected for this study was to generate representative distributions using Monte Carlo simulations of a wide range of inverse Gaussian distributions and, then, to evaluate the functions numerically. This allows conclusions about the estimators based on on their performance with simulated data and comparisons to existing theory.

The data for the highly skewed distribution cases indicates that the rejection rate for the non-null distributions is virtually the same as for the null distributions. This behavior can be expected because of the large variability in the data for those distributions. To achieve the accuracy that is required to distinguish differences in means requires much larger sample sizes than was selected for comparison.

This situation tends to gradually change as the distributions become less skewed. At first the restricted maximum likelihood estimator (RMLE) tends to dominate the other estimators. Then, the efficiency of the RMLE drops off rapidly. Concurrently, the data shows that increasingly higher rejection rates of the null hypothesis occurs, which is the desired behavior for any non-null distribution.

As the distributions become more normally distributed, the shrinkage estimator (SE) begins to dominate the PTE, especially for larger differences from the null and for larger sample sizes.

Previous conclusions about regions of relative dominance of the estimators were based strictly on theoretical analysis for the normal distribution case. In general, the risk and efficiency data for a wide range of Inverse Gaussian distributions tends to validate those conclusions. This is the most significant result of the investigation.

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