AN INVESTIGATION OF THE EQUILIBRIUM POINTS OF THE RESTRICTED THREE BODY PROBLEM IN AN INERTIAL COORDINATE SYSTEM

by

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ABSTRACT

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The equilibrium points of the restricted three body problem were investigated and numerically evaluated utilizing equations derived within an inertial coordinate system. The three body system investigated was an Earth - Moon-Satellite system. The masses, m_1, m_2, and m_3, were defined as the earth, moon, and satellite, respectively, and the origin of the inertial coordinates was located at the center of the earth. The differential equations of motion for the satellite were analyzed within the non-rotating coordinate system, and a set of equations were derived to define the equilateral and collinear equilibrium points. The equations for the collinear points were solved numerically on a computer and then compared to the results obtained using equations for the collinear points derived within a rotating coordinate system. The origin of the rotating system was the center of mass of the system. The numerical results agreed. The equilateral points were shown to agree by analysis.

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INTRODUCTION AND BACKGROUND

Over the years numerous individuals have devoted considerable effort investigating the restricted three body problem. Many studies analyze the motion of the third body in relation to the other two. Others have looked at interesting features of the mechanics of the three body system as a whole. Szebehely thoroughly investigates many aspects of the restricted three body problem from zero velocity curves to Hill's problem. (*Szebehely* 1967, 159-200) An in depth discussion of the restricted three body problem could easily consume an entire academic year. One interesting feature of the restricted three body problem is the presence of the equilibrium points, unique locations where the relative motion of the three bodies is zero. The existence of these points was first proven by the mathematician Lagrange, while Caratheodory also provided a proof for the existence of the equilibrium points. (*Sommerfeld* 1964, 176) Rather than attempt an alternative proof for the existence of these points or even solve the restricted three body problem itself, an interesting study is to examine the equilibrium points and attempt to determine their locations using a new set of equations. Utilizing equations derived in an inertial coordinate system and comparing the results to those obtained with equations derived in a rotating coordinate system, an investigation of the equilibrium points of the restricted three body problem in an inertial coordinate system follows.

Statement of the Problem

The restricted three body problem is defined as follows: Two bodies revolve in circular orbits around their center of mass influenced by their mutual gravitational attraction. A third body which is attracted by the first two but does not significantly influence their motion moves in the plane of motion of the revolving bodies. The restricted problem of three bodies is to describe the motion of this third body. (*Szebehely* 1967, 17-18) The two revolving bodies are called the primaries. Their masses are considered point masses, labeled m_1 and m_2. The third body, m_3, is much less massive than the primaries. For this study, m_1, m_2, and m_3, are defined as the earth, moon, and satellite, respectively. In order to investigate the restricted three body problem, certain assumptions, or approximations, are made. Some of these assumptions serve to focus attention on the three body system and to exclude external factors, while others are approximations within the system itself. One assumption is that there are no external perturbations to the bodies. In an Earth - Moon - Satellite system, this assumption disregards the perturbing effects of other planets and the sun on the motion of the three bodies. The first assumption made within the three body system itself is that m_3 is infinitesimal. This implies that m_1 and m_2 influence the motion of m_3, but m_3 exerts no force on m_1 or m_2. This follows, since in the definition of the restricted three body problem, m_3 is stated not to significantly influence the motion of m_1 or m_2. The second assumption is that m_1 and m_2 are in circular orbits about their center of mass. This approximation prevents variations in the distance between m_1 and m_2. As a result of these assumptions, and since the motion being studied is that of a satellite in relation to the earth and the moon, a geocentric, non-rotating coordinate system is chosen to be a valid approximation of an inertial coordinate system.

In the three body system there exist five distinct points at which the relative motion of m_3 to the m_1, m_2 pair is zero. These are called the equilibrium points and are further defined as the equilateral and collinear equilibrium points due to their relative locations (Figure 1). The equilateral points form two equilateral triangles with the primaries while the collinear points lie along the line connecting m_1 and m_2. This fact is essential to the continuation of the study. The existence of these points will not be proven, but rather, knowing these points to exist, two methods for determining their locations will be compared.

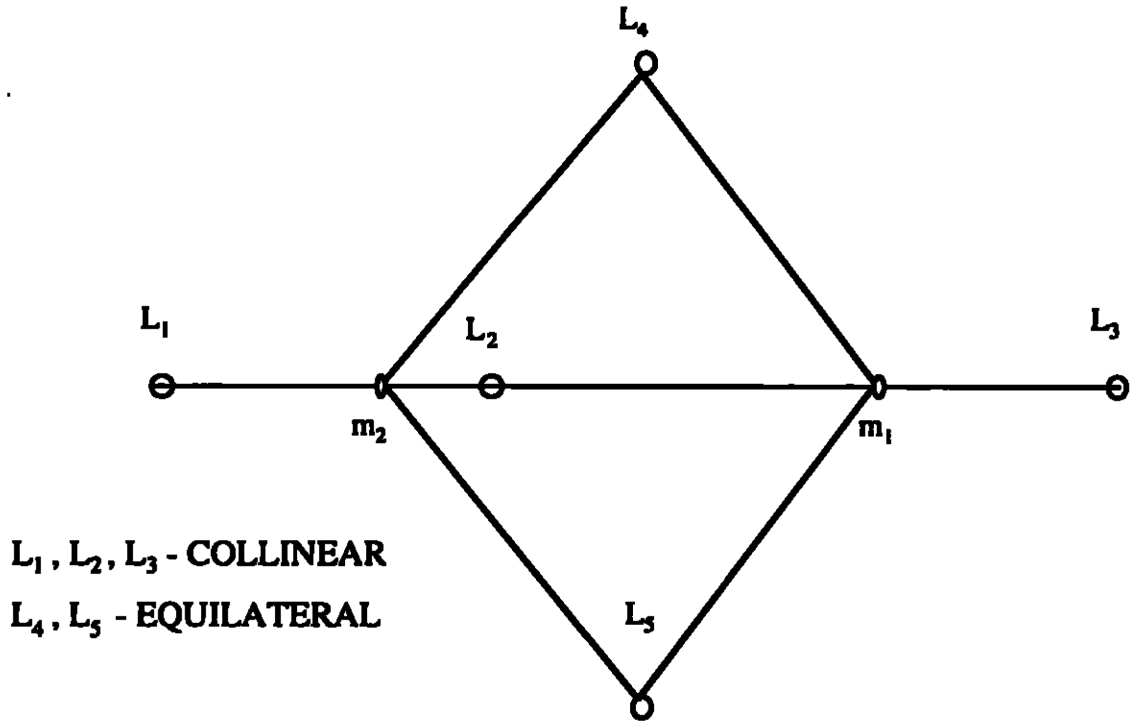

Figure 1 Location of Equilibrium Points in Relation to the Primaries

The equilibrium points are labeled L_1 through L_5 after the 18th century mathematician Lagrange who contributed greatly to the initial study of these points. It was Lagrange who in 1772 predicted the presence of the Trojan asteroids at the L_4 and L_5 points of the Sun - Jupiter - Satellite system. It was not until 1906, when observational equipment had matured sufficiently to detect these asteroids, that his prediction was confirmed. (*Szebehely* 1967, 278-280)

Background

An investigation of the restricted three body problem typically begins in a non-rotating coordinate system with its origin located at the center of mass. The problem is traditionally transformed into a rotating, or synodic, coordinate system. This transformation is done to remove the dependence on time from the gravitational force function which defines the motion of the primaries. In a non-rotating coordinate system (Figure 2), the time dependence of the coordinates for m_1 and m_2 introduces time explicitly into the equations of motion for m_3. (*Szebehely* 1967, 23)

Szebehely shows that transforming the problem into a rotating system (Figure 2) removes the explicit dependence on time from the coordinates for m_1 and m_2 and therefore from the equations of motion for m_3. In a coordinate system which rotates with the same angular velocity as m_1 and m_2, the primaries appear fixed. This results in equations of motion which depend solely on the initial conditions and can therefore predict the motion of m_3. (*Szebehely* 1967, 29) The equilibrium points of the restricted three body problem also rotate with the same angular velocity of the primaries about the center of mass, and therefore appear stationary as well.

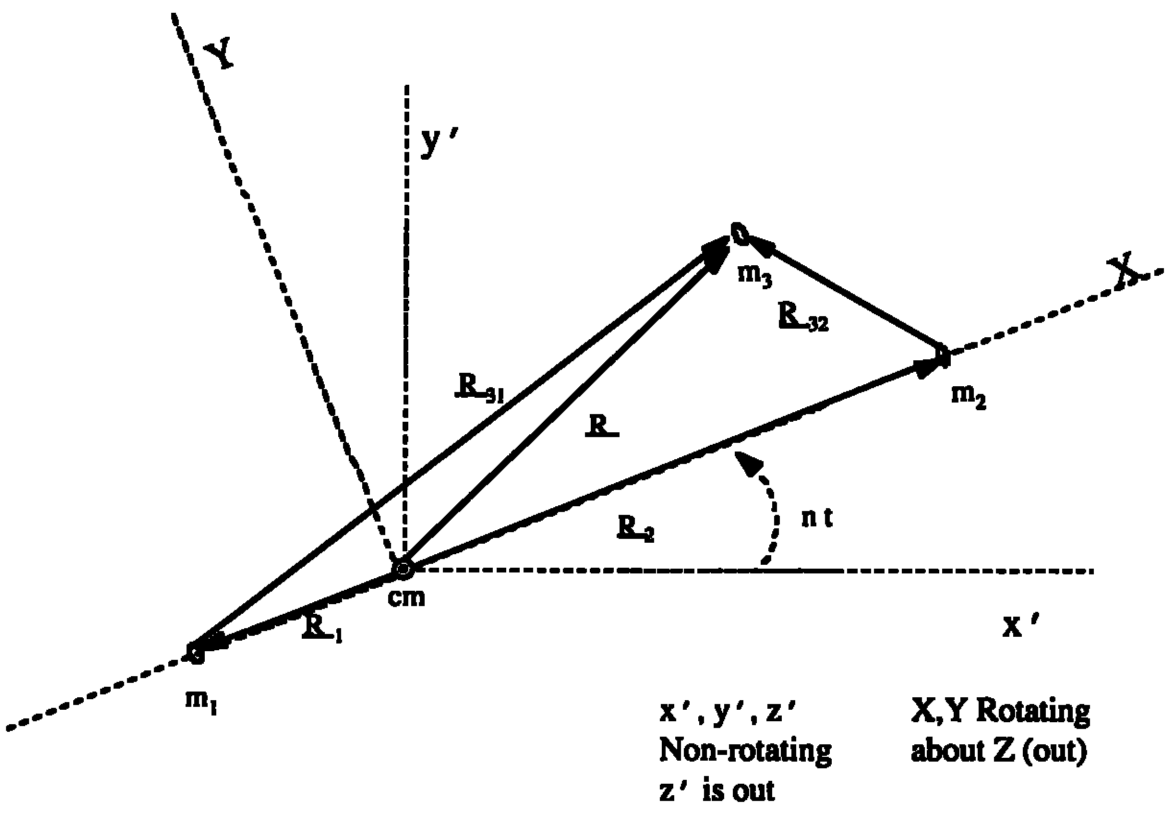

Figure 2 Restricted Three Body Problem in Rotating Coordinates

Before beginning with the discussion of the equilibrium point calculations, an overview of the method for obtaining the differential equations of motion in the rotating system is warranted. A more detailed discussion is presented in Appendix 1. For clarity, the notation to be used is now described. Vector quantities are represented as underlined variables while scalar quantities are not underlined. Three coordinate systems will be considered, a rotating system with origin at the center of mass, a non-rotating system with origin at the center of mass, and a non-rotating system with origin at m_1. The coordinate systems are labeled as XYZ, $x'y'z'$, xyz respectively. The xyz system is the approximation of the inertial system. Consequently, the vector from the origin to m_2 in the rotating system would be represented as $\underline{R}_2$, while the magnitude of the vector from m_2 to m_3 in the inertial system would be represented as r_{32}.

The traditional approach to the restricted three body problem is to begin in a non-rotating system with equations for the mutual gravitational forces of the three bodies. Since the center of mass of the system is not accelerating, it can be chosen as the origin of the coordinate system. (*Bond* 1997, 2) The assumptions made to the three body system then begin to simplify the equations. The assumption that m_3 is infinitesimal removes the terms which include m_3 from the equations for the accelerations on m_1 and m_2. The assumption that m_1 and m_2 are in circular orbits causes the magnitude of the vectors $\underline{R}_1$ and $\underline{R}_2$ to be constant. The mean motion of the m_1, m_2 system is constant as well due to this same assumption. The mean motion of the system, n, is the angular rate of the moon about the earth and is further described in Appendix 1.

It is at this point that the problem is typically transformed to a rotating coordinate system by the relation

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = A \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

where

$$A = \begin{pmatrix} \cos nt & -\sin nt & 0 \\ \sin nt & \cos nt & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and n is the mean motion of the m_1, m_2 system. (*Szebehely* 1967, 121-122) The mean motion can also be related to the period of m_2 by the following equation.

$$n = \frac{2\pi}{P}$$

Here the equations contain variables of distance and time, and the technique of non-dimensionalization is used. In non-dimensionalization, all distances are scaled by R_{12}, the constant m_1, m_2 distance, and time is scaled by the mean motion of the two body system formed by the primaries. (*Szebehely* 1967, 16-25) Two quantities, μ_1 and μ_2, are also defined.

$$\mu_1 = \frac{m_1}{m_1 + m_2}$$

and

$$\mu_2 = \frac{m_2}{m_1 + m_2}$$

μ_1 and μ_2 are the non-dimensionalized masses of m_1 and m_2 respectively. Therefore by definition, $\mu_1 + \mu_2 = 1$. This places the entire problem in non-dimensional coordinates. Now the problem can be shifted to a more convenient coordinate system. In this coordinate system, the origin is still at the center of mass, but the larger mass, m_1 , is placed to the right of the origin, and the smaller mass, m_2 , is placed to the left. Since the distances have been scaled by R_{12} , the distance between m_1 and m_2 is now 1. μ_2 is also redefined as μ, which leads to $\mu_1 = 1 - \mu$. The coordinates of the points are P_1 (μ , 0) and P_2 (μ - 1 , 0) where m_1 and m_2 are located at P_1 and P_2 respectively.

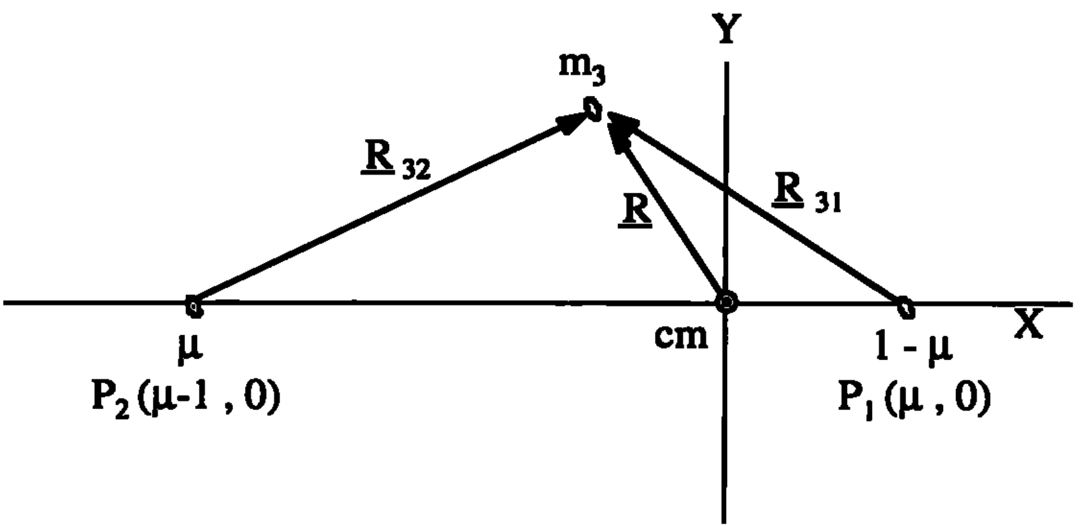

Figure 3 Coordinate System with Large Mass on the Right

The non-dimensionalized equations are simplified and then integrated to obtain the differential equations of motion of m_3 with respect to m_1 and m_2 . As part of the simplification process prior to integration, a force function, Ω, is defined. A discussion of Ω is included in Appendix 1. The final result is that the differential equations of motion for m_3 in a rotating coordinate system are as follows:

$$\ddot{X} - 2\dot{Y} = \Omega_X = X - \frac{\mu_1}{R_1^3}(X - \mu_2) - \frac{\mu_2}{R_2^3}(X + \mu_1)$$

$$\ddot{Y} + 2\dot{X} = \Omega_Y = Y - \frac{\mu_1}{R_1^3}Y - \frac{\mu_2}{R_2^3}Y$$

$$\ddot{Z} = \Omega_Z = -\frac{\mu_1}{R_1^3}Z - \frac{\mu_2}{R_2^3}Z$$

where

$$\Omega = \frac{1}{2}\mu_1\mu_2 + \frac{1}{2}(X^2 + Y^2) + \frac{\mu_1}{R_1} + \frac{\mu_2}{R_2}$$

$$R_1^2 = (X - \mu_2)^2 + Y^2 + Z^2$$

$$R_2^2 = (X + \mu_1)^2 + Y^2 + Z^2$$

The function Ω is a potential, and the differential equations of motion for m_3 above show the partial derivatives of Ω with respect to X, Y, and Z.

$$\frac{\partial \Omega}{\partial X} = \Omega_X$$

$$\frac{\partial \Omega}{\partial Y} = \Omega_Y$$

$$\frac{\partial \Omega}{\partial Z} = \Omega_Z$$

Statement of the Work

This is an investigation of the locations of the equilibrium points and not an attempt to solve the restricted three body problem. The evaluation of the equilibrium points is done at time $t = t_0$. The time t_0 was selected such that the horizontal axes of the inertial and the rotating systems are coincident. This results in the y coordinate for m_2 in the inertial system being equal to zero. The goal of the investigation is to determine the location of the equilibrium points of an Earth - Moon - Satellite system utilizing equations derived in an inertial coordinate system. Although the position of the equilibrium points will vary explicitly with time, since they revolve at the same angular rate as the m_1 , m_2 pair, they will remain stationary in relation to the primaries. These results are then compared to the location of the equilibrium points calculated utilizing equations derived in a rotating system.

DETERMINATION OF THE EQUILIBRIUM POINTS

Derivation and Analysis in the Rotating Coordinate System

The equilibrium points of the restricted three body problem have no motion with respect to the rotating system. This means

$$X = \text{constant}_1$$
$$Y = \text{constant}_2$$
$$Z = \text{constant}_3$$

is a solution to the differential equations of motion. The locations of the equilibrium points are found when the differential equations of motion, (Ω_X, Ω_Y, Ω_Z) are set equal to zero. At these values of X, Y, and Z the relative motion of m_3 with respect to m_1 and m_2 is zero. The differential equations of motion for m_3 in the rotating system were presented in the introduction. Since the equilibrium condition exists when the solution components, X, Y, Z, are constant, then the velocity and acceleration components are individually zero as well.

Therefore, from the differential equations previously, the equilibrium condition exists when the following is true

$$\Omega_X = 0$$

$$\Omega_Y = 0$$

$$\Omega_Z = 0.$$

Beginning with the equation for Ω_Z and factoring out (-Z) we obtain

$$\Omega_z = -Z\left(\frac{\mu_1}{R_1^3} + \frac{\mu_2}{R_2^3}\right) = 0$$

The quantity $\left(\frac{\mu_1}{R_1^3} + \frac{\mu_2}{R_2^3}\right)$ is greater than zero. Therefore, in order for Ω_z to be equal to zero, Z must be equal to zero. This means that the equilibrium points will lie in the XY plane which is the plane of motion for m_1 and m_2.

Choosing the coordinate system with the large mass on the right and $\mu_2 = \mu$ and $\mu_1 = 1 - \mu$. Substituting into the equations for Ω_x and Ω_y yields the following equations:

$$\Omega_X = X - \frac{(1-\mu)}{R_1^3}(X-\mu) - \frac{\mu}{R_2^3}(X+1-\mu)$$

$$\Omega_Y = Y\left(1 - \frac{1-\mu}{R_1^3} - \frac{\mu}{R_2^3}\right)$$

where

$$R_1^2 = (X-\mu)^2 + Y^2$$

$$R_2^2 = (X+1-\mu)^2 + Y^2. \qquad (\textit{Bond}\ 1997, 6)$$

By inspection, one solution is that Ω_x and Ω_y are equal to zero when R_1 and R_2 are equal to one. To find the X and Y values which satisfy this condition, we first subtract the equation for R_2^2 from the equation for R_1^2 to obtain

$$(X-\mu)^2-(X+1-\mu)^2=0$$

which is satisfied when X is equal to (μ - 1/2). Substituting this value for X into the equations for R_1^2 and R_2^2 results in

$$Y=\pm\frac{\sqrt{3}}{2} \ .$$

This results in the equilateral equilibrium points which form two equilateral triangles with the primaries. (Figure 4) The equilateral equilibrium points are labeled L_4 and L_5 . Interpreting these results in the non-dimensionalized coordinates shows that the X coordinate lies one half the distance between m_1 and m_2 , at (μ - 1/2) , and the Y coordinates are equal to $\pm$ 0.8660 times the distance between the primaries.

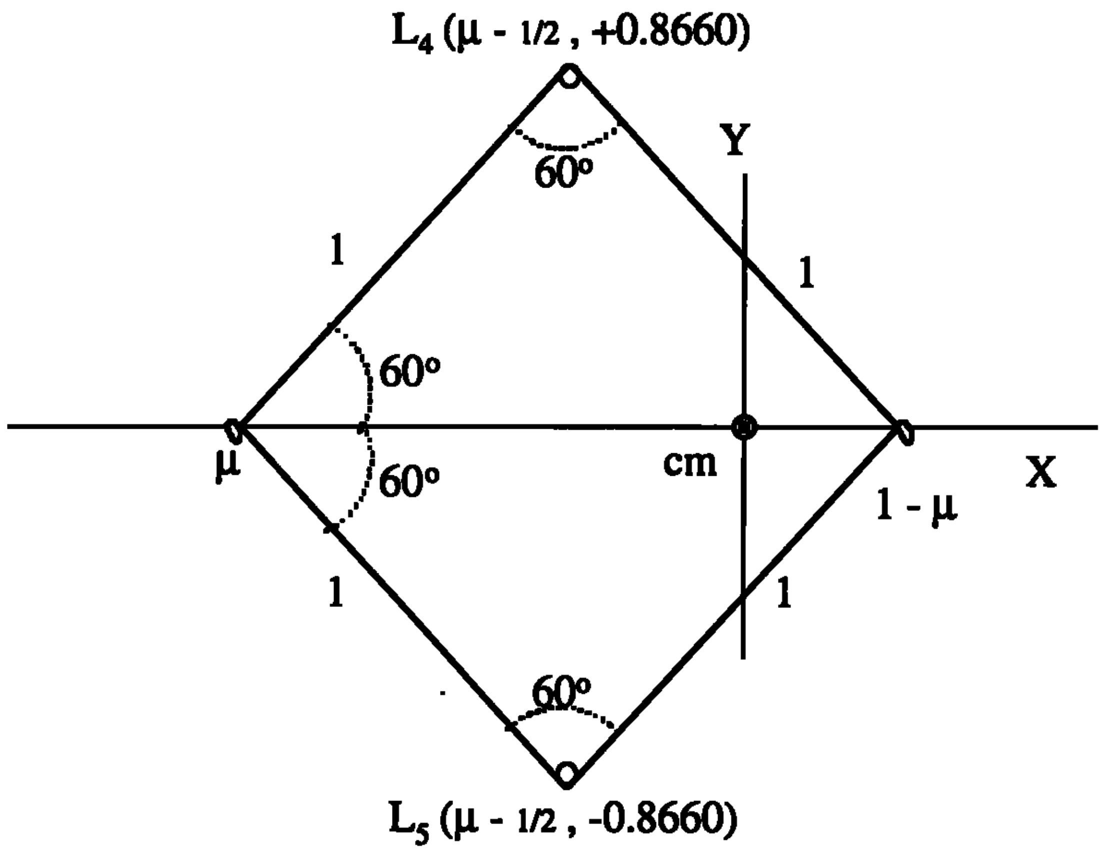

Figure 4 Equilateral Points in Non-Dimensional Coordinates

Further inspection of the differential equations of motion shows that Ω_Y will also be equal to zero when Y is equal to zero. This means that the remaining equilibrium points will lie along the line joining m_1 and m_2. These points are called the collinear equilibrium points. To obtain a value for X we substitute Y equal to zero into the equations for R_1^2 and R_2^2 to obtain

$$X - \mu = \pm R_1$$
$$X - \mu + 1 = \pm R_2$$

This results in an equation for Ω_X which when set equal to zero and solved produces the X values for the remaining equilibrium points. Taking into account the possible sign combinations the equation can be written as follows:

$$\Omega_X = X + k_1 \frac{1-\mu}{(X-\mu)^2} + k_2 \frac{\mu}{(X+1-\mu)^2} = 0$$

where

$$k_1 = \pm 1 \text{ and } k_2 = \pm 1.$$

There are four possible combinations of the signs for k_1 and k_2, each corresponding to a value for R_1 and R_2. By analysis, a relative positioning of the equilibrium points can be found. (Figure 5) The possible sign combinations are as follows:

<u>Case I</u> $k_1 = 1$ $k_2 = 1$

$X - \mu > 0$ and $X + 1 - \mu > 0$. In this case, X will be greater than μ, and X will be greater than $\mu - 1$. Since μ and μ-1 are the X coordinates of the primaries, this means that the equilibrium point will lie to the right of both m_1 and m_2. This point is labeled L_3.

<u>Case II</u> $k_1 = 1$ $k_2 = -1$

$X - \mu > 0$ and $X + 1 - \mu < 0$. In this case, X would be greater than μ, and X would also be less than $\mu - 1$, but this presents a problem. If $X - \mu > 0$ is true then $(X + 1 - \mu)$ is also a positive number. Thus, if X is greater than μ, $X + 1 - \mu < 0$ can not be true. This contradiction does not lead to a real solution for this case.

<u>Case III</u> $k_1 = -1$ $k_2 = 1$

$X - \mu < 0$ and $X + 1 - \mu > 0$. In this case, X will be less than μ, and X will be greater than $\mu - 1$. This results in the equilibrium point being located to the left of m_1 and to the right of m_2. This point is labeled L_2.

<u>Case IV</u> $k_1 = -1$ $k_2 = -1$

$X - \mu < 0$ and $X + 1 - \mu < 0$. In this case, X will be less than μ, and X will be less than $\mu - 1$. This means that the point will lie to the left of both m_1 and m_2. This point is labeled L_1. (*Bond* 1997, 11)

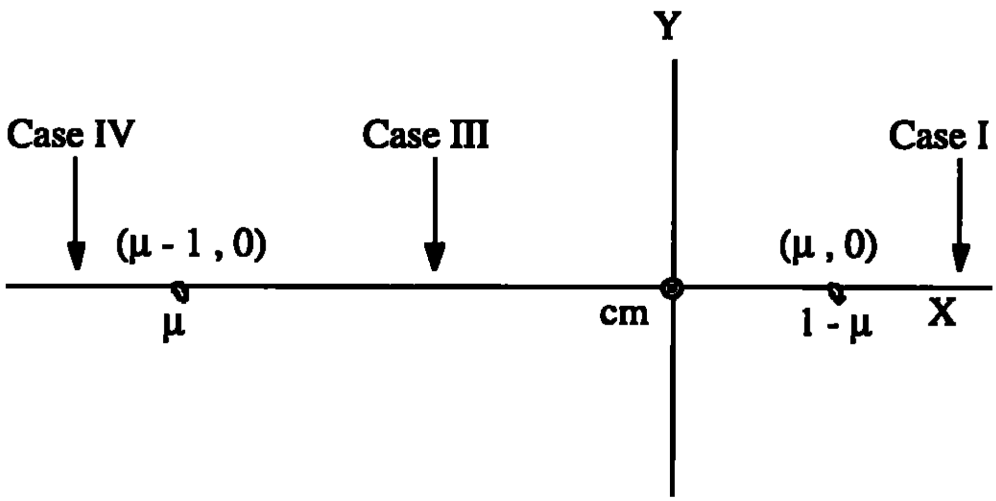

Figure 5 Collinear Equilibrium Points

Derivation and Analysis in the Inertial Coordinate System

Referring to Figure 6, the differential equation of motion of m_3 with respect to m_1 is shown as equation (1). From this point, certain equations will be numbered for more convenient reference.

$$\ddot{\underline{r}} + Gm_1 \frac{\underline{r}}{r^3} = -Gm_2 \left(\frac{\underline{r}_{32}}{r_{32}^3} + \frac{\underline{r}_2}{r_2^3} \right) \tag{1}$$

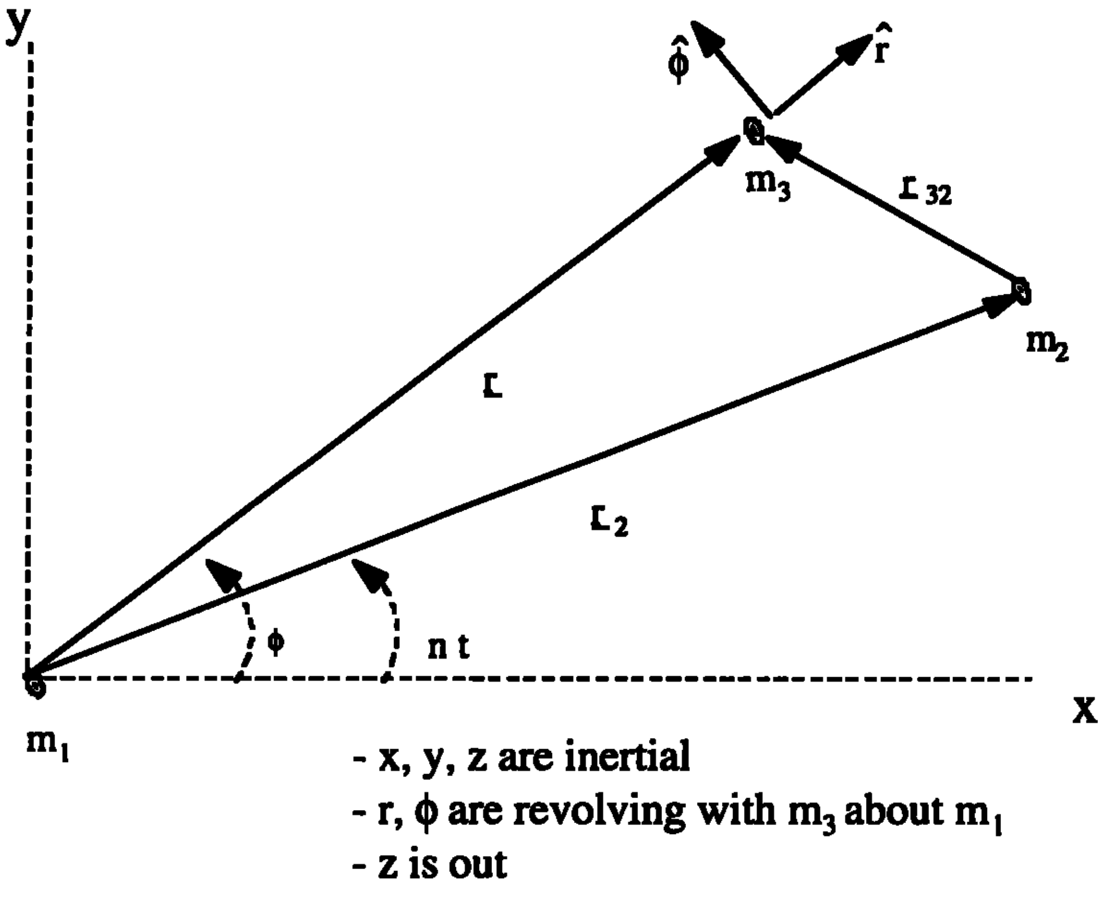

Figure 6 Restricted Three Body Problem in Inertial Coordinates

Based on the previous assumptions made in the analysis of the restricted three body problem, the mass of m_3 approaches zero, $m_3 \to 0$, and the m_1, m_2 distance is constant, r_2 = constant. (*Bond* 1997, 22) In this system, the motion of the primaries is such that the distance from the origin (the location of m_1) to m_2 is equal to the mean distance between the earth and the moon. Thus, m_1 is fixed at the origin while m_2 and m_3 move.

Also from Figure 6, the relationship of the vectors can be shown to be

$$\underline{r}_{32} = \underline{r} - \underline{r}_2 ,$$

and the constant angular velocity of the system to be

$$\underline{\omega} = \omega\hat{k},$$

where

$$\omega^2 = \frac{G(m_1 + m_2)}{r_2^3}. \qquad \text{(\textit{Bond and Allman} 1993, 165-166)}$$

Since r_2 is equal to R_{12}, ω in the inertial system is equal to n from the rotating system. This follows since we are describing the angular rate of the same system in different coordinates.

Equation (2) is the acceleration on m_3 in the $\hat{r}$, $\hat{\phi}$ system

$$\ddot{\underline{r}} = \hat{r}\left(\ddot{r} - r\dot{\phi}^2\right) + \hat{\phi}\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\phi}\right). \qquad \text{(2) (\textit{Danby} 1962, 31-32)}$$

To begin the investigation of the equilibrium points, we set equations (1) and (2) equal to one another and obtain

$$\hat{r}\left(\ddot{r} - r\dot{\phi}^2\right) + \hat{\phi}\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\phi}\right) = -Gm_1\frac{\underline{r}}{r^3} - Gm_2\left(\frac{\underline{r}_{32}}{r_{32}^3} + \frac{\underline{r}_2}{r_2^3}\right)$$

Next we find the scalar product of the above equation with $\hat{r}$ and again separately with $\hat{\phi}$ to obtain equations (3) and (4).

$$\ddot{r} - r\dot{\phi}^2 = -\frac{Gm_1}{r^2} - Gm_2\left(\frac{\hat{r}\cdot\underline{r}_{32}}{r_{32}^3} + \frac{\hat{r}\cdot\underline{r}_2}{r_2^3}\right) \tag{3}$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\phi}\right) = -Gm_2\left(\frac{\hat{\phi}\cdot\underline{r}_{32}}{r_{32}^3} + \frac{\hat{\phi}\cdot\underline{r}_2}{r_2^3}\right) \tag{4}$$

At this juncture the equilateral points can be examined. Since the solution in the rotating system is known, assume those results here in the inertial system to be true and set $r = r_2 = r_{32}$.

We now investigate the solutions to equations (3) and (4) if these assumptions are true.

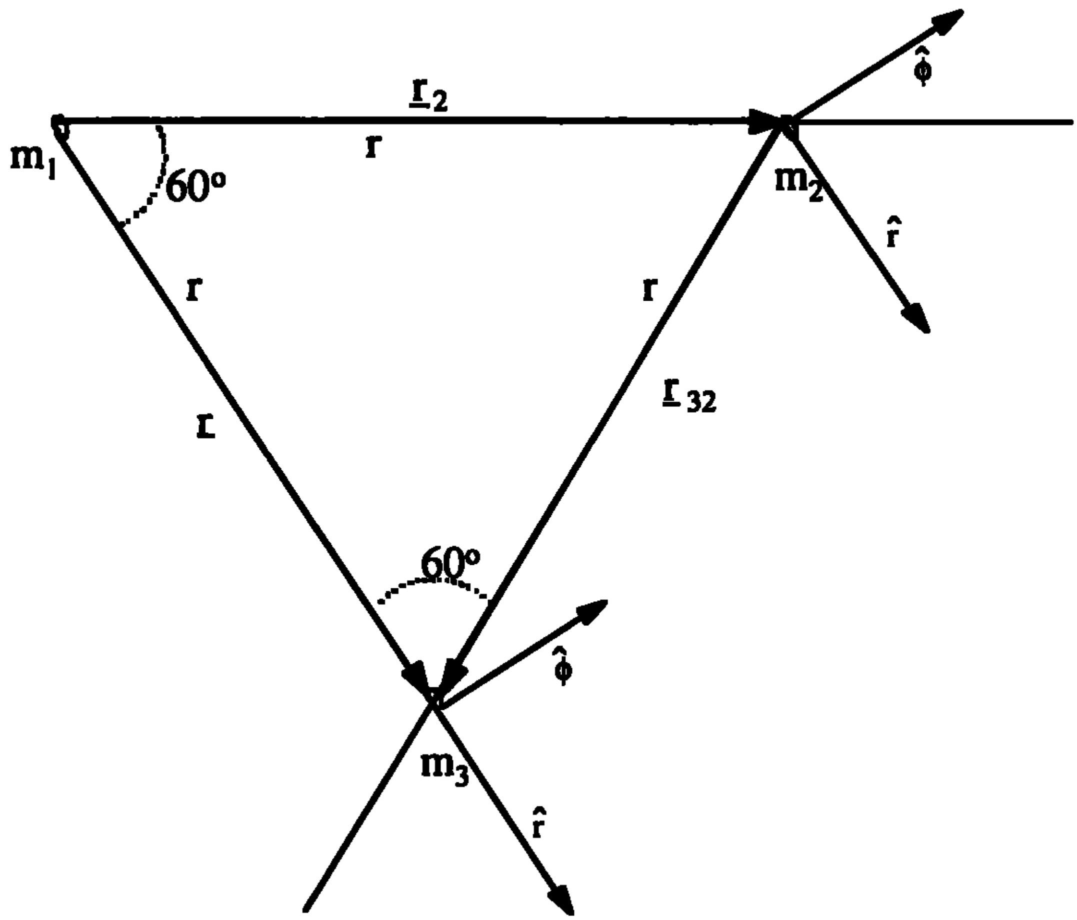

Figure 7   Equilateral Equilibrium Points in Inertial Coordinates

From Figure 7 the following values can be shown:

$$\hat{\phi} \cdot \underline{r}_{32} = r_{32} \cos (60^\circ + 90^\circ) = -\frac{\sqrt{3}}{2} r_2$$

$$\hat{\phi} \cdot \underline{r}_2 = r_2 \cos (30^\circ) = \frac{\sqrt{3}}{2} r_2$$

$$\hat{r} \cdot \underline{r}_{32} = r_{32} \cos (60^\circ) = \frac{1}{2} r_2$$

$$\hat{r} \cdot \underline{r}_2 = r_2 \cos (60^\circ) = \frac{1}{2} r_2$$

Substituting into equations (3) and (4) we obtain

$$-r_2\dot{\phi}^2 = -Gm_2\left(\frac{r_2}{2r_2^3}+\frac{r_2}{2r_2^3}\right)-\frac{Gm_1}{r_2^2} = -\frac{Gm_2}{r_2^2}-\frac{Gm_1}{r_2^2} \qquad (5)$$

$$r_2\ddot{\phi} = -Gm_2\left(\frac{-\sqrt{3}}{2}\frac{r_2}{r_2^3}+\frac{\sqrt{3}}{2}\frac{r_2}{r_2^3}\right) = 0 \qquad (6)$$

Equation (6) shows that $\dot{\phi}$ = constant. Using this result in equation (5) yields

$$\dot{\phi}^2 = \frac{G(m_1+m_2)}{r_2^3} = \text{constant}\ .$$

Referring back to the initial assumptions in the inertial system,

$$\omega^2 = \frac{G(m_1+m_2)}{r_2^3}$$

so $\dot{\phi}^2 = \omega^2$. Since $\dot{\phi}$ is the constant angular rate of the m_1, m_3 pair, and it is equal to ω, which is the constant angular rate for the m_1, m_2 pair, the points are all moving at the same angular rate and the geometry of the initial assumption is preserved as being valid. Therefore, the equations which support this geometry are valid as well. (*Bond* 1997, 26) In his proof of the existence of the equilateral equilibrium points, Caratheodory uses similar logic regarding the preservation of geometry. (*Sommerfeld* 1964, 176-80)

Since the locations of the collinear points are restricted to the line connecting m_1 and m_2, $\underline{r}_{32}$ will either be directed along $\underline{r}$ or 180° opposite. The same relationship is true for $\underline{r}_{32}$ and $\underline{r}_2$. Adopting the same convention as in the rotating system, let

$$\underline{r}_{32} = k_1\, r_{32}\, \hat{r}$$

$$\underline{r}_2 = k_2\, r_2\, \hat{r}$$

where $k_1 = \pm 1$ and $k_2 = \pm 1$.

With these relationships, note the following to be true:

$$\hat{r} \cdot \underline{r}_{32} = k_1\, r_{32}$$

$$\hat{r} \cdot \underline{r}_2 = k_2\, r_2$$

$$\hat{\phi} \cdot \underline{r}_{32} = 0$$

$$\hat{\phi} \cdot \underline{r}_2 = 0 \ .$$

Substituting the above values into equations (3) and (4), equation (3) becomes

$$\ddot{r} - r\dot{\phi}^2 = -\frac{Gm_1}{r^2} - Gm_2\left(\frac{k_1}{r_{32}^2} + \frac{k_2}{r_2^2}\right) \qquad (7)$$

and equation (4) becomes

$$\frac{d}{dt}\left(r^2\dot{\phi}\right) = 0 \qquad (8)$$

Similar to the discussion of the equilateral points, for m_3 to remain aligned with the m_1, m_2 pair, the angular rate for m_1, m_3 must be the same as that for m_1, m_2.

Since $\dot{\phi}^2 = \omega^2 =$ constant, that means

$$\dot{\phi}\frac{d}{dt}r^2 = 0$$

therefore,

$$r^2 = \text{constant},$$

which in turn leads to

$$\dot{r} = 0,$$

$$\ddot{r} = 0.$$

Substituting into equation (7) above we obtain

$$-r\frac{G(m_1 + m_2)}{r_2^3} = -\frac{Gm_1}{r^2} - Gm_2\left(\frac{k_1}{r_{32}^2} + \frac{k_2}{r_2^2}\right) \tag{9}$$

In an attempt to obtain an equation which looks familiar, reverse the signs and multiply by r_2^2. The result is

$$r\frac{G(m_1 + m_2)}{r_2} = r_2^2\frac{Gm_1}{r^2} - Gm_2\left(\frac{k_1 r_2^2}{r_{32}^2} + k_2\right). \tag{10}$$

Divide by G ($m_1 + m_2$)

$$\frac{r}{r_2} = \frac{r_2^2}{r^2}\frac{m_1}{m_1+m_2} + \frac{m_2}{m_1+m_2}\left(\frac{k_1 r_2^2}{r_{32}^2} + k_2\right). \tag{11}$$

Substitute for μ_1 and μ_2

$$\frac{r}{r_2} = \frac{r_2^2}{r^2}\mu_1 + \mu_2\left(\frac{k_1 r_2^2}{r_{32}^2} + k_2\right). \tag{12}$$

If we again shift to a coordinate system with the large mass on the right, recalling that $\mu_1 = 1 - \mu$ and $\mu_2 = \mu$, equation (12) becomes

$$\frac{r}{r_2} = \frac{r_2^2}{r^2}(1-\mu) + \mu\left(\frac{k_1 r_2^2}{r_{32}^2} + k_2\right) \tag{13}$$

To accurately reflect that the origin of the coordinate system is now located at the center of mass, the axes will be renamed x' and y'.

To evaluate the equation, set it equal to zero and note r_2 in each term

$$\frac{r}{r_2} - (1-\mu)\frac{1}{r^2 / r_2^2} - \mu\left(\frac{k_1}{r_{32}^2 / r_2^2} + k_2\right) = 0 \tag{14}$$

To simplify the equation, substitute for r / r_2 and r_{32} / r_2 by defining two new variables, Δ and δ where

$$\Delta = r / r_2$$

and

$$\delta = r_{32} / r_2 .$$

Equation (14) becomes

$$\Delta - \frac{(1-\mu)}{\Delta^2} - \mu\left(\frac{k_1}{\delta^2} + k_2\right) = 0 \qquad (15)$$

Since $\underline{r}_{32}$ is the vector from m_2 to m_3 , this is the value which we want to find. Investigating r_{32}^2 yields:

$$r_{32}^2 = (\underline{r} - \underline{r}_2) \cdot (\underline{r} - \underline{r}_2) = r^2 + r_2^2 - 2\, \underline{r} \cdot r_2$$

$$r_{32}^2 = r^2 + r_2^2 - 2\, k_2\, r\, r_2$$

$$r_{32}^2 / r_2^2 = (r^2 / r_2^2) + 1 - 2\, k_2\, (r / r_2) \quad . \qquad (16)$$

Utilizing the definitions of Δ and δ, equation (14) becomes

$$\delta^2 = \Delta^2 + 1 - 2\, k_2\, \Delta . \qquad (17)$$

In an attempt to simplify equation (15) above, multiply by Δ^2

$$\Delta^3 - (1-\mu) - \mu\left(\frac{k_1\Delta^2}{\delta^2} + k_2\Delta^2\right) = 0 ;$$

next, multiply through by δ^2

$$\Delta^3 \delta^2 - (1 - \mu)\, \delta^2 - \mu\, k_1\, \Delta^2 - \mu\, k_2\, \Delta^2\, \delta^2 = 0 \;;$$

$$\Delta^3 \delta^2 - \delta^2 + \mu\, \delta^2 - \mu\, k_1\, \Delta^2 - \mu\, k_2\, \Delta^2\, \delta^2 = 0 \;; \qquad (18)$$

substitute equation (17) into equation (18)

$$\Delta^3 (\Delta^2 + 1 - 2\, k_2\, \Delta) - (\Delta^2 + 1 - 2\, k_2\, \Delta) + \mu\, (\Delta^2 + 1 - 2\, k_2\, \Delta)$$
$$- \mu\, k_1\, \Delta^2 - \mu\, k_2\, \Delta^2 (\Delta^2 + 1 - 2\, k_2\, \Delta) = 0 \;;$$

distribute the terms in parentheses

$$\Delta^5 + \Delta^3 - 2\, k_2\, \Delta^4 - \Delta^2 - 1 + 2\, k_2\, \Delta + \mu\, \Delta^2 + \mu$$
$$- 2\, k_2\, \mu\, \Delta - \mu\, k_1\, \Delta^2 - \mu\, k_2\, \Delta^2 (\Delta^2 + 1 - 2\, k_2\, \Delta) = 0 \;;$$

$$\Delta^5 + \Delta^3 - 2\, k_2\, \Delta^4 - \Delta^2 - 1 + 2\, k_2\, \Delta + \mu\, \Delta^2 + \mu - 2\, k_2\, \mu\, \Delta$$
$$- \mu\, k_1\, \Delta^2 - \mu\, k_2\, \Delta^4 - \mu k_2\, \Delta^2 + 2\, \mu\, k_2^2\, \Delta^3 = 0 \;;$$

collect coefficients for powers of Δ

$$\Delta^5 + \Delta^4 (- 2\, k_2 - \mu\, k_2) + \Delta^3 (1 + 2\, \mu\, k_2^2) + \Delta^2 (-1 + \mu - \mu\, k_1 - \mu\, k_2)$$
$$+ \Delta\, (2\, k_2 - 2\, \mu\, k_2) + \mu - 1 = 0 \;;$$

and simplify to obtain equation (19)

$$\Delta^5 + k_2\, (2 + \mu)\, \Delta^4 + (1 + 2\, \mu)\, \Delta^3 - (1 - \mu + \mu\, k_1 + \mu\, k_2)\, \Delta^2 + 2\, k_2\, (1 - \mu)\, \Delta + \mu - 1 = 0 \,.$$

(19)

The values for Δ found from equation (19) will be used to determine the x' coordinates of the collinear equilibrium points.

As was the case for the equilateral points, the equations must be evaluated with the possible sign combinations for k_1 and k_2 . There are four possible combinations of the signs for k_1 and k_2, each corresponding to a value for $\underline{r}_2$ and $\underline{r}_{32}$. Utilizing the coordinate system with the large mass on the right, a relative positioning of the equilibrium points can be found through analysis. The possible sign combinations are as follows:

<u>Case I</u> $k_1 = 1$ $k_2 = 1$

Since $\underline{r}$ defines the direction of $\hat{r}$, if k_1 and k_2 are both positive, $\underline{r}$, $\underline{r}_2$, and $\underline{r}_{32}$ will all point in the same direction. This will place the equilibrium point to the left of both m_1 and m_2 at L_1 . (Figure 8)

<u>Case II</u> $k_1 = 1$ $k_2 = -1$

In this case $\underline{r}_{32}$ is in the same direction as $\underline{r}$, and opposite to $\underline{r}_2$. This will place the equilibrium point to the right of m_2 and also to the right of the center of mass. However, analysis can not determine in which position, L_2 or L_3, relative to m_1 the point will lie. Numerical results for the magnitude of $\underline{r}$ will show this point to be located at L_3 .

<u>Case III</u> $k_1 = -1$ $k_2 = 1$

In this case $\underline{r}_2$ is in the same direction as $\underline{r}$, and opposite to $\underline{r}_{32}$ which means the equilibrium point will lie in between m_1 and m_2 at L_2.

<u>Case IV</u> $k_1 = -1$ $k_2 = -1$

For this case to be true, both $\underline{r}_2$ and $\underline{r}_{32}$ would have to be opposite to $\underline{r}$. Since $\underline{r}$ determines the direction of $\hat{r}$, it is impossible for both $\underline{r}_2$ and $\underline{r}_{32}$ to point in the same direction and then have $\underline{r}$ point in the opposite direction. This contradiction does not lead to a real solution for this case.

Figure 8 displays the results of the case analysis. The axes labels x' and y' reflect the coordinate system shift which placed the center of mass at the origin. Upon completion of the numerical calculations this coordinate shift will need to be taken into account. This analysis will prove to be important in the evaluation of the numerical computations. Since $\Delta = r/r_2$, the roots of the equation for Δ will only provide the magnitude of the vector $\underline{r}$. The direction is determined by this case analysis.

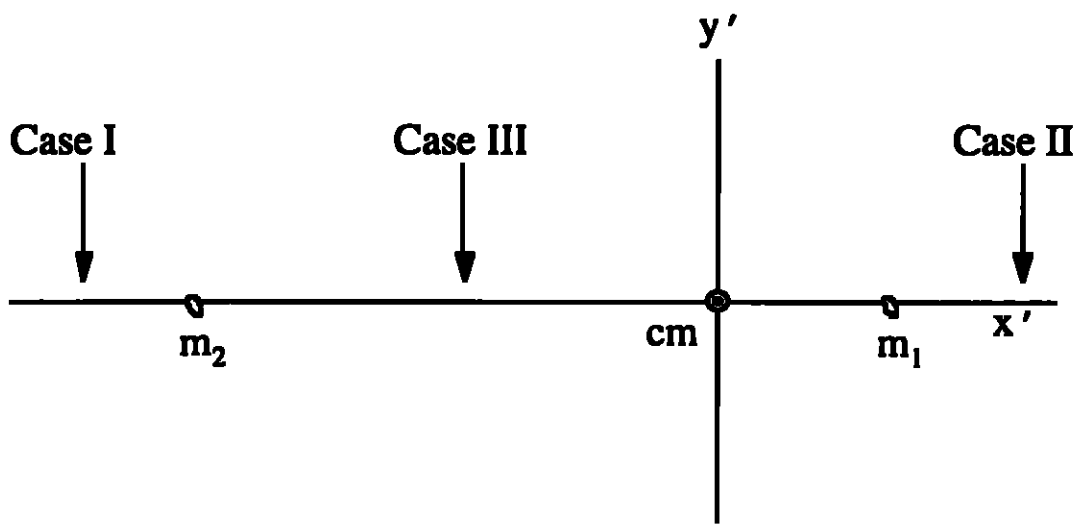

Figure 8 Collinear Equilibrium Points in Non-Rotating Coordinates

Numerical Computations

The capabilities of the Mathematica® software to numerically solve for the roots of polynomial equations were used for the computer calculations. Mathematica is a registered trademark of Wolfram Research Incorporated. (*Wolfram*, 1994) The procedure was similar for the calculations in both the inertial and rotating systems. First, the constants were defined. The mass of the earth and the mass of the moon were entered as m_1 and m_2 . Then the non-dimensionalized masses were calculated. Since the coordinate system with the large mass on the right was used, μ_2 was defined to be equal to μ. Next the equation to be solved was entered and given the symbol "eqn". In the inertial system, "eqn" is a fifth degree polynomial in delta. A series of Mathematica commands was given to set the equation equal to zero and solve for delta assigning each of the possible sign combinations of k_1 and k_2 . The Mathematica® commands and their results for the inertial system are copied below. In Mathematica®, input commands are printed in bold type while output is printed in plain type.

```
m1 = 5.9722 * 10^24
          24
5.9722 10

m2 = 7.3458 * 10^22
          22
7.3458 10

mu1 = m1 / (m1 + m2)
0.987849

mu2 = m2 / (m1 + m2)
0.0121505

mu = mu2
0.0121505

eqn = delta^5 - k2 (2 + mu) delta^4 + (1 + 2 mu)
delta^3 - (1 - mu + mu k1 + mu k2) delta^2
 + 2 k2 (1 - mu) delta + mu - 1

Solve [eqn==0, delta] /.{k1 -> 1, k2 -> 1}
 {{delta -> -0.493239 - 0.86366 I},

  {delta -> -0.493239 + 0.86366 I},

  {delta -> 0.915398 - 0.131042 I},

  {delta -> 0.915398 + 0.131042 I},

  {delta -> 1.16783}}
```

```
Solve [eqn==0, delta] /.{k1 -> 1, k2 -> -1}
 {{delta -> -0.998482 - 0.0787135 I},
  {delta -> -0.998482 + 0.0787135 I},
  {delta -> -0.504049 - 0.858894 I},
  {delta -> -0.504049 + 0.858894 I},
  {delta -> 0.992912}}

Solve [eqn==0, delta] /.{k1 -> -1, k2 -> 1}
 {{delta -> -0.494611 - 0.861321 I},
  {delta -> -0.494611 + 0.861321 I},
  {delta -> 1.07615 - 0.145786 I},
  {delta -> 1.07615 + 0.145786 I},
  {delta -> 0.849066}}
```

In the rotating system, the procedure was similar. The same variables were defined, and the equation for the X coordinates of the collinear equilibrium points was assigned the same "eqn" symbol. Once again, a series of Mathematica commands was given to set the equation equal to zero, and this time solve for X, with each of the possible sign combinations for k_1 and k_2. The Mathematica commands and their results for the rotating system are copied below.

```
m1 = 5.9722 * 10^24

          24
5.9722 10

m2 = 7.3458 * 10^22

          22
7.3458 10

mu1 = m1 / (m1 + m2)

0.987849

mu2 = m2 / (m1 + m2)

0.0121505

mu = mu2

0.0121505

eqn = x + k1 * ((1 - mu) / (x - mu)^2)
+ k2 * (mu / (x + 1 - mu)^2)

Solve[eqn==0, x] /.{k1 -> 1, k2 -> 1}
 {{x -> -1.15568},

  {x -> -0.903247 - 0.131042 I},

  {x -> -0.903247 + 0.131042 I},

  {x -> 0.505389 - 0.86366 I},

  {x -> 0.505389 + 0.86366 I}}

Solve[eqn==0, x] /.{k1 -> 1, k2 -> -1}
 {{x -> -1.064 - 0.145786 I},

  {x -> -1.064 + 0.145786 I},

  {x -> -0.836915},

  {x -> 0.506761 - 0.861321 I},

  {x -> 0.506761 + 0.861321 I}}
```

```
Solve[eqn==0, x] /.{k1 -> -1, k2 -> -1}
 {{x -> -0.986331 - 0.0787135 I},
  {x -> -0.986331 + 0.0787135 I},
  {x -> -0.491899 - 0.858894 I},
  {x -> -0.491899 + 0.858894 I},
  {x -> 1.00506}}
```

INTERPRETATION OF THE STUDY

Analysis and Comparison of Numerical Results

As shown in the Mathematica output copied above, each case in both the inertial and rotating system generated five solutions to its equation, one real and four imaginary. The solutions which contain imaginary numbers are not considered to be physical solutions and are therefore not discussed. For the same reason, the case in each coordinate system which did not represent a real solution is not discussed as well.

In order to determine the location of the collinear equilibrium points in the inertial system, the Mathematica results for Δ must be converted to a value for r. This is done by referring to the definition of Δ .

$$\Delta = r / r_2$$

Multiplying the Mathematica results by r_2 will yield the distance to each of the collinear points. This distance, r , is the magnitude of the vector $\underline{r}$, while the direction of $\underline{r}$ is determined by each one of the case analyses previously performed on the possible sign combinations for k_1 and k_2 .

The determination of the location of the collinear points in the rotating system is more straightforward. The Mathematica output for x is the X coordinate for the collinear point in the non-dimensionalized system. Since R_{12} was the scaling factor used in the non-dimensionalization process, multiplying the results by R_{12} would convert the results back to dimensional coordinates.

For a comparison of the results, the Mathematica output could be converted to dimensional coordinates and evaluated, however, there is a simpler comparative approach. Since R_{12} and r_2 are both equal to the mean distance from the earth to the moon, and both sets of results must be multiplied by that value for dimensional comparison, the Mathematica output itself can be compared directly and viewed as fractional parts of the earth, moon distance. Upon initial inspection of the Mathematica output, the two results may not seem to concur. (Table 1) After a brief investigation, any apparent disagreement can be easily explained.

Table 1 Comparison of Calculated Values for L_1, L_2, and L_3

	L_1	L_2	L_3
Inertial	1.16783	0.84907	0.99291
Rotating	-1.15568	-0.83692	1.00506
Delta	0.01215	0.01215	0.01215

In each case the absolute values, the magnitudes of the vectors $\underline{r}$ and $\underline{R}$ differ by the same amount. The inertial number would seem to displace the equilibrium point to the left by a value of 0.01215. This number, 0.01215, is the calculated value for μ, the non-dimensionalized mass of m_2. Recall that in the formulation of the equations in the inertial system, upon introducing the non-dimensionalized masses, we shifted to the more convenient x', y' z' coordinate system which placed the large mass on the right and the center of mass at the origin. This shift in the position of m_1 was exactly μ. It is this shift which accounts for the delta. It was stated previously that this shift would need to be taken into account. Since t_0 was selected such that the x, x', and X axes were all coincident, the transformation is elementary.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x'+\mu \\ y' \\ z' \end{pmatrix}$$

Accounting for this fact results in agreement in the calculated values for the collinear equilibrium points in both the inertial and rotating systems.

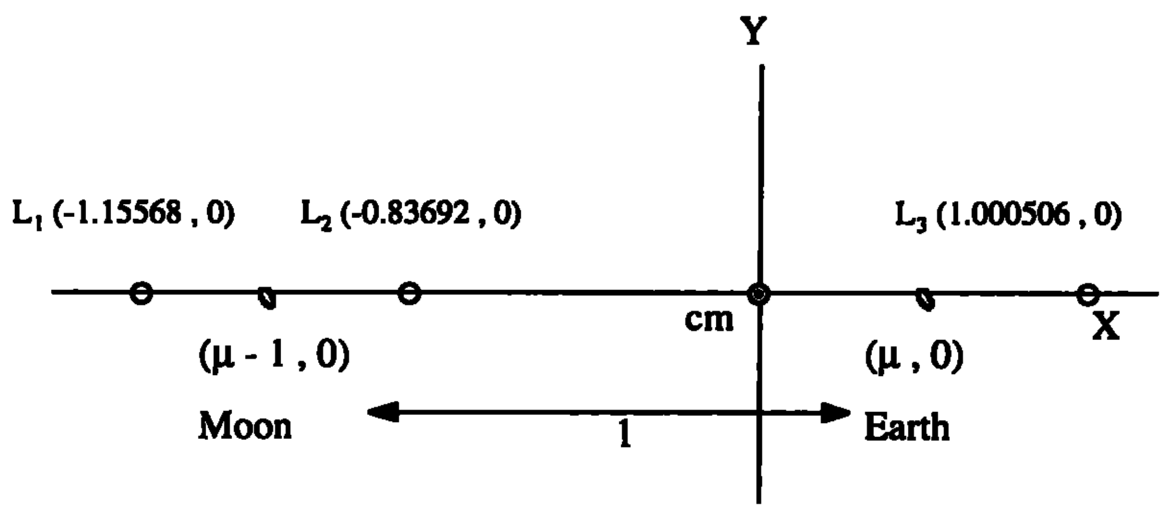

Figure 9 Collinear Points in Non-Dimensional Coordinates (not to scale)

The physical interpretation of the results in the rotating system is that the collinear equilibrium points lie at positions expressed as relative percentages of the earth, moon distance. L_1 is positioned 15.568% of the earth, moon distance on the far side of the moon. L_2 is located 83.692% of the way from the earth to the moon. L_3 is 100.504% of the earth, moon distance or approximately the earth, moon distance on the side of the earth opposite the moon.

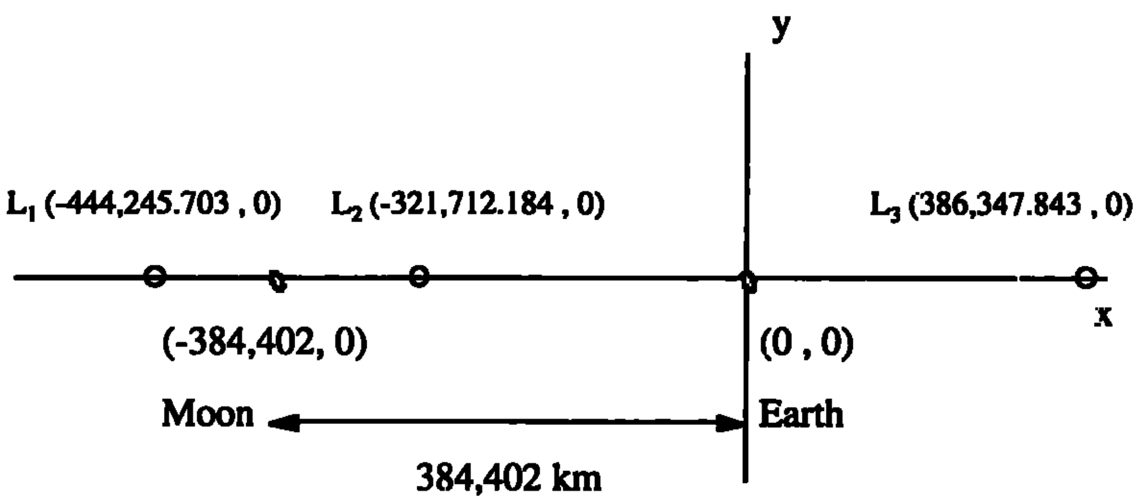

Figure 10 Collinear Points in Dimensional Coordinates (not to scale)

The physical results of the computations for the collinear equilibrium points in dimensional coordinates are shown in Figure 10 above. The physical results of the computations for the equilateral equilibrium points in dimensional coordinates are shown in Figure 11 below.

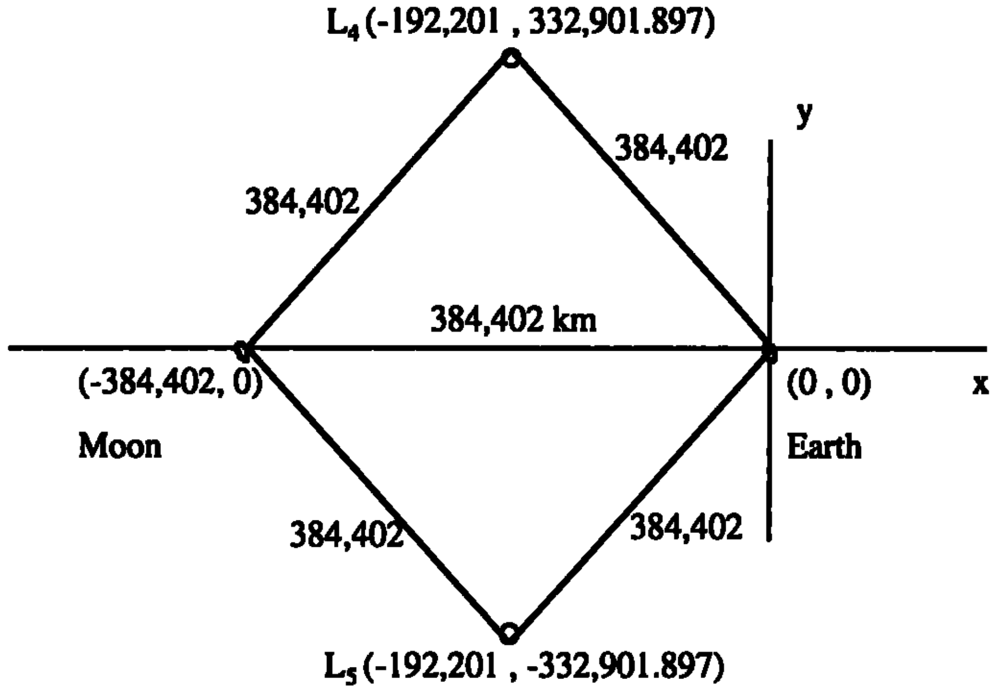

Figure 11 Equilateral Points in Dimensional Coordinates

Since the variables in the Mathematica program are the masses of the primaries, it should be possible to extend these equations to other three body systems which hold the same assumptions. By substituting the masses of the primaries of another three body system, the location of the collinear equilibrium points should be attainable.

Summary

The equilibrium points of the restricted three body problem were investigated in a geocentric, non-rotating, coordinate system which served as an approximation of an inertial coordinate system. Equations for determining the position of the equilibrium points in an inertial coordinate system were derived. The locations of the equilibrium points for the restricted three body problem were calculated in both an inertial and a rotating coordinate system. The equilateral equilibrium points were shown to agree by analysis. This analysis showed that when substituting the known solution from the rotating system into the equations derived in the inertial system, the geometry of the equilateral points was preserved. The locations of the collinear points were calculated via computer from equations derived in both the inertial and rotating systems. Imaginary solutions were not considered. The physical results were shown to agree. This study was successful in showing that while there are a well known set of equations for determining the location of the equilibrium points of the restricted three body problem, an alternative set of equations does exist. These alternative equations do generate accurate results.

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APPENDIX 1

A more detailed discussion of three topics presented in the study is shown below. These comments are primarily a combination from Szebehely and Bond, and so being a blend of methods are not referenced by individual steps. For a more in depth discussion of these topics, see Szebehely.

Mean Motion

In the background discussion of the restricted three body problem in rotating coordinates, reference is made to the mean motion of the m_1, m_2 system. The mean motion is the angular rate of rotation of the m_1, m_2 pair, and therefore the magnitude of the angular velocity. The mean motion is defined as follows:

$$n = \sqrt{\frac{G^2(m_1 + m_2)}{R_{12}^3}} \, .$$

Since m_1 and m_2 are assumed to be in circular orbits, R_{12} is constant. This means that n is constant as well. It is this assumption which allows n to be used as a scaling factor in the non-dimensionalization process.

As the discussion proceeds into the inertial coordinate system, reference is made to the angular velocity, $\underline{\omega}$. The magnitude of $\underline{\omega}$, is defined as

$$\omega = \sqrt{\frac{G^2(m_1 + m_2)}{r_2^3}}$$

Since R_{12} and r_2 are both equal to the mean distance between the earth and the moon, it is apparent from the equations that n and ω are equal. During the evaluation of the equilateral points in the inertial system, this value ω was also shown to be equal to $\dot{\phi}$. The rate of change of the angle ϕ will only be equal to n and ω in the equilibrium condition.

The Force Function Ω

During the development of the differential equations of motion in the rotating system, a force function Ω was defined.

$$\Omega = \frac{1}{2}\mu_1\mu_2 + \frac{1}{2}(X^2 + Y^2) + \left(\frac{\mu_1}{R_1}\right) - \left(\frac{\mu_2}{R_2}\right)$$

$$R_1^2 = (X - \mu_2)^2 + Y^2 + Z^2$$

$$R_2^2 = (X + \mu_1)^2 + Y^2 + Z^2$$

and

$$\Omega_x = X - \frac{\mu_1}{R_1^3}(X - \mu_2) - \frac{\mu_2}{R_2^3}(X + \mu_1)$$

$$\Omega_y = Y - \frac{\mu_1}{R_1^3}Y - \frac{\mu_2}{R_2^3}Y$$

$$\Omega_z = -\frac{\mu_1}{R_1^3}Z - \frac{\mu_2}{R_2^3}Z$$

The force function is combination of a gravitational force as well as the centripetal force of the system. The value of Ω is based upon the initial position of m_3 in the system. The partial derivatives of Ω with respect to X, Y, and Z, are dependent upon the initial conditions for X, Y, and Z respectively.

Equations of Motion in the Rotating System

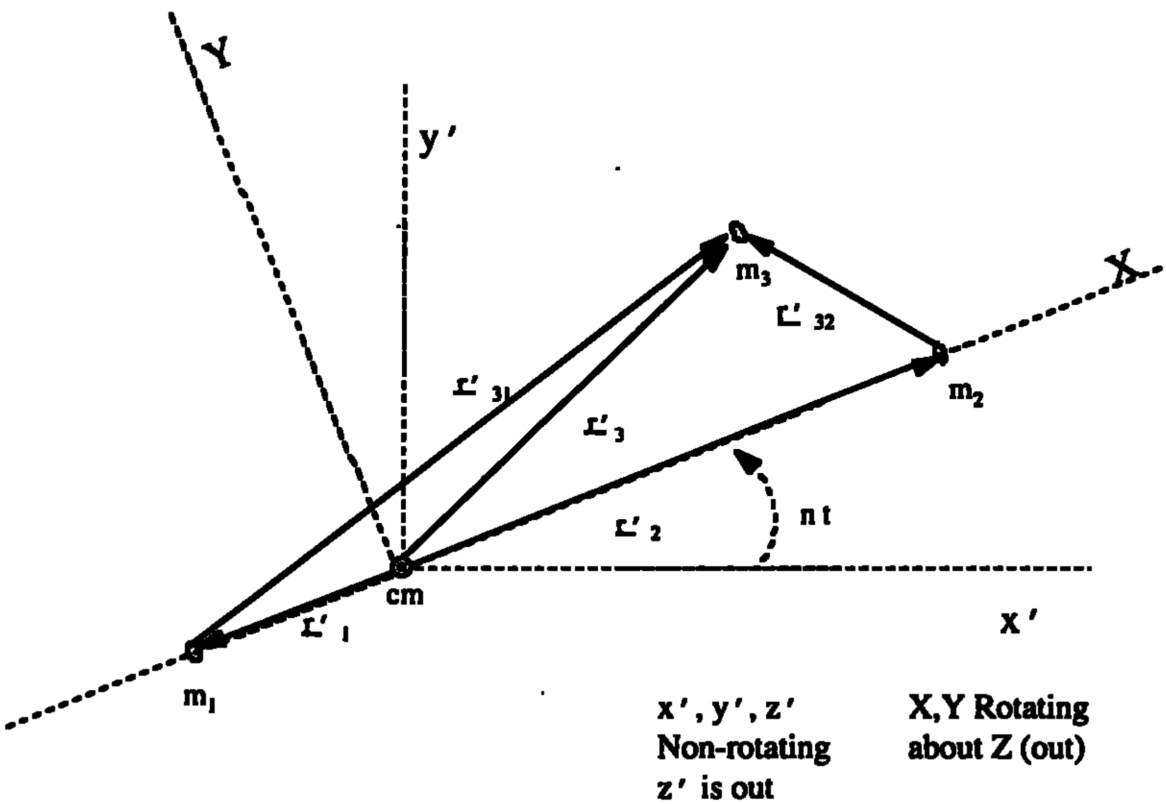

Figure 12 Restricted Three Body Problem in Non-rotating Coordinates

Referring to Figure 12, begin with the equations for the accelerations on each of the bodies.

$$\ddot{\underline{r}}'_1 = -k^2 m_2 \frac{\underline{r}'_1 - \underline{r}'_2}{r'^3_{12}} - k^2 m_3 \frac{\underline{r}'_1 - \underline{r}'_3}{r^3_{13}}$$

$$\ddot{\underline{r}}'_2 = -k^2 m_1 \frac{\underline{r}'_2 - \underline{r}'_1}{r'^3_{12}} - k^2 m_3 \frac{\underline{r}'_2 - \underline{r}'_3}{r'^3_{23}}$$

$$\ddot{\underline{r}}'_3 = -k^2 m_1 \frac{\underline{r}'_3 - \underline{r}'_1}{r'^3_{13}} - k^2 m_2 \frac{\underline{r}'_3 - \underline{r}'_2}{r'^3_{32}}$$

Now the assumption that m_3 is infinitesimal simplifies the equations as shown

$$\ddot{\underline{r}}'_1 = -k^2 m_2 \frac{\underline{r}'_1 - \underline{r}'_2}{r'^3_{12}}$$

$$\ddot{\underline{r}}'_2 = -k^2 m_1 \frac{\underline{r}'_2 - \underline{r}'_1}{r'^3_{12}}$$

$$\ddot{\underline{r}}'_3 = -k^2 m_1 \frac{\underline{r}'_3 - \underline{r}'_1}{r'^3_{13}} - k^2 m_2 \frac{\underline{r}'_3 - \underline{r}'_2}{r'^3_{32}}$$

Recall that $\underline{r}'_{21} = \underline{r}'_2 - \underline{r}'_1$ and subtract the equation for $\underline{r}'_1$ from $\underline{r}'_2$ to obtain

$$\ddot{\underline{r}}'_{21} = \ddot{\underline{r}}'_2 - \ddot{\underline{r}}'_1 = -k^2 m_1 \frac{\underline{r}'_2 - \underline{r}'_1}{r'^3_{12}} + k^2 m_2 \frac{\underline{r}'_1 - \underline{r}'_2}{r'^3_{12}}$$

$$\ddot{\underline{r}}'_{21} = -k^2 m_1 \frac{\underline{r}'_{21}}{r'^3_{12}} - k^2 m_2 \frac{\underline{r}'_{21}}{r'^3_{12}}$$

$$\ddot{\underline{r}}'_{21} = -k^2 (m_1 + m_2) \frac{\underline{r}'_{21}}{r'^3_{12}}$$

$$\ddot{\underline{r}}'_{21} + \frac{k^2(m_1+m_2)}{r'^3_{12}}\underline{r}'_{21} = 0 \tag{20}$$

and

$$\ddot{\underline{r}}'_3 = -k^2 m_1 \frac{\underline{r}'_3 - \underline{r}'_1}{r'^3_{13}} - k^2 m_2 \frac{\underline{r}'_3 - \underline{r}'_2}{r'^3_{32}} \tag{21}$$

Equation (20) is the two body motion of m_1 and m_2 while equation (21) is the motion of m_3 with respect to the center of mass. This is the point at which the problem is generally transformed into the rotating system.

Since the center of mass is the origin of the system

$$m_1 r_1' = m_2 r_2'$$
$$r_{12}' = r_1' + r_2'$$

so

$$m_1 r_1' = m_2(r_{12}' - r_1')$$
$$(m_1 + m_2) r_1' = m_2 r_{12}'$$

and

$$r_1' = \frac{m_2}{m_1 + m_2} r_{12}' = \mu_2 r_{12}'$$

$$r_2' = \frac{m_1}{m_1 + m_2} r_{12}' = \mu_1 r_{12}'$$

The right hand side of equation (21) can be written in the rotating system as follows

$$\mathrm{RHS} = -k^2 m_1 \frac{\underline{R}_{31}}{R_{31}^3} - k^2 m_2 \frac{\underline{R}_{32}}{R_{32}^3} \tag{22}$$

and the right hand side of equation (21) can be written in the rotating system by use of the following transformation

$$\ddot{\underline{r}}'_3 = A\underline{R}$$

where

$$A = \begin{pmatrix} \cos nt & -\sin nt & 0 \\ \sin nt & \cos nt & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

From here we develop the equations for $\dot{r}'_3$ and $\ddot{\underline{r}}'_3$ by taking the derivative of $\underline{r}'_3$.

$$\dot{\underline{r}}_3 = \dot{\underline{R}} + n \times \underline{R}$$

$$\ddot{\underline{r}}'_3 = \ddot{\underline{R}} + 2\underline{n} \times \dot{\underline{R}} + \underline{n} \times (\underline{n} \times \underline{R}) \tag{23}$$

where

$$\underline{n} = n\hat{k} \ .$$

Equations (22) and (23) can be set equal as the right and left hand sides of equation (21)

$$\ddot{\underline{R}} + 2\underline{n} \times \dot{\underline{R}} + \underline{n} \times (\underline{n} \times \underline{R}) = -k^2 m_1 \frac{R_{31}}{R_{31}^3} - k^2 m_2 \frac{R_{32}}{R_{32}^3} \tag{24}$$

and after evaluating the cross products, equation (24) can be written in column notation as

$$\begin{pmatrix} \ddot{X} \\ \ddot{Y} \\ \ddot{Z} \end{pmatrix} + 2n \begin{pmatrix} -\dot{Y} \\ \dot{X} \\ 0 \end{pmatrix} - n^2 \begin{pmatrix} X \\ Y \\ 0 \end{pmatrix} = \frac{-k^2 m_1}{R_1^3} \begin{pmatrix} X - \mu_2 r'_{12} \\ Y \\ Z \end{pmatrix} - \frac{k^2 m_2}{R_2^3} \begin{pmatrix} X + \mu_1 r'_{12} \\ Y \\ Z \end{pmatrix} \tag{25}$$

where

$$R_1^2 = (X - \mu_2 r'_{12})^2 + Y^2 + Z^2$$
$$R_2^2 = (X + \mu_1 r'_{12})^2 + Y^2 + Z^2 \qquad (26)\ (27)$$

Two quantities are defined as d_1 and d_2.

$$d_1 = \frac{R_1}{r'_{12}}$$
$$d_2 = \frac{R_2}{r'_{12}}$$

Next the non-dimensionalization process is begun by scaling the distances by the m1, m_2, distance and time by the mean motion. The scaling of time is in fact a multiplication process where

$$\tau = nt$$

and τ is fictitious time. Non-dimensionalization is completed by the substitution of the values for d_1, d_2, μ_1, and μ_2. Equations (28), (29), and (30) show the non-dimensionalized form of equation (25) by component.

$$X'' - 2Y' = X - \frac{\mu_1}{d_1^3}(X - \mu_2) - \frac{\mu_2}{d_2^3}(X + \mu_1) \qquad (28)$$

$$Y'' + 2X' = Y - \frac{\mu_1}{d_1^3}Y - \frac{\mu_2}{d_2^3}Y \qquad (29)$$

$$Z'' = -\frac{\mu_1}{d_1^3}Z - \frac{\mu_2}{d_2^3}Z \qquad (30)$$

Equations (26) and (27) become

$$d_1^2 = (X - \mu_2)^2 + Y^2 + Z^2 \qquad (31)$$

$$d_2^2 = (X + \mu_1)^2 + Y^2 + Z^2 \qquad (32)$$

It is at this point that the force function Ω is introduced as

$$\Omega = \frac{1}{2}\mu_1\mu_2 + \frac{1}{2}\left(X^2 + Y^2\right) + \frac{\mu_1}{d_1} + \frac{\mu_2}{d_2} \tag{33}$$

Ω can be used to reduce equations (28), (29), and (30) by taking the partial derivative of Ω with respect to X, Y, and Z.

$$\Omega_X = X - \frac{\mu_1}{d_1^2}\frac{\partial d_1}{\partial x} - \frac{\mu_2}{d_2^2}\frac{\partial d_2}{\partial x} \tag{34}$$

$$\Omega_Y = Y - \frac{\mu_1}{d_1^2}\frac{\partial d_1}{\partial y} - \frac{\mu_2}{d_2^2}\frac{\partial d_2}{\partial y} \tag{35}$$

$$\Omega_Z = -\frac{\mu_1}{d_1^2}\frac{\partial d_1}{\partial z} - \frac{\mu_2}{d_2^2}\frac{\partial d_2}{\partial z} \tag{36}$$

Taking the partial derivatives of (31) and (32) and substituting those into (34), (35), and (36) we obtain

$$\Omega_X = X - \frac{\mu_1}{d_1^3}(X - \mu_2) - \frac{\mu_2}{d_2^3}(X + \mu_1) \tag{37}$$

$$\Omega_Y = Y - \frac{\mu_1}{d_1^3}Y - \frac{\mu_2}{d_2^3}Y \tag{38}$$

$$\Omega_Z = -\frac{\mu_1}{d_1^3}Z - \frac{\mu_2}{d_2^3}Z \tag{39}$$

which are equal to the right hands side of (28), (29), and (30). So it is shown that

$$X'' - 2Y' = \Omega_X$$
$$Y'' + 2X' = \Omega_Y$$
$$Z'' = \Omega_Z$$