LORENTZ INVARIANCE IN LOOP QUANTUM GRAVITY

by

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Dedication

To my friends and family who have supported me along the way, and to the future generations of scientists who will be attempting to discover the very nature of reality.

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ABSTRACT

LORENTZ INVARIANCE IN LOOP QUANTUM GRAVITY

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From the principle of relativity, where the laws of physics is presumed to be the same in all inertial reference frames, we have Lorentz invariance. This invariance leads to rotational and boost invariance. These invariances are most recognizable in the theory of special relativity. For decades, physicists have been attempting to develop a theory of quantum gravity where all of general relativity, special relativity, and quantum mechanics can come together. Two of the biggest developments so far have been that of String Theory and Loop Quantum Gravity. In the case of Loop Quantum Gravity, Lorentz invariance has not emerged so smoothly. This is because of the postulates of Loop Quantum Gravity, postulating that a discrete structure of spacetime exists near the Planck scale, where there is a minimum length and minimum time. The minimum length and minimum time are the Planck length and Planck time, respectively. The crucial role of the Planck scale is that it is the scale at which gravitational effects become relevant in a quantum setting, unlike in quantum mechanics where gravitational effects are too weak to play much of a role. The minimum length, however, appears to be a contradiction to that of Lorentz invariance since a boost in one frame of reference would lead to a length contraction. By considering relative locality and the postulates of Loop Quantum Gravity, we develop two ways in which we could resolve this contradiction.

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CHAPTER I:
INTRODUCTION

A significant problem in the two major fields of quantum mechanics and general relativity is that they do not merge together consistently. This is a problem considering that both fields describe the same universe, although at different scales. Quantum mechanics describes the universe at the smallest scales while general relativity describes the universe at much larger scales up to the size of galaxies and much more. Since both exists in the same universe, it is logical to conclude that at some point the two should merge and be able to accurately describe the same object or the same system. This is one of the reasons for the need of a theory on quantum gravity. Another reason is that a successful theory of quantum gravity would help explain various problems in cosmology, such as describing quantum effects during the early universe or on black holes where gravitational forces are strong. Such a theory would also be a major step in unifying the four fundamental forces, which would eventually lead to a Grand Unifying Theory and a Theory on Everything. However, for decades physicists have failed to come up with a successful solution. Currently, there are at least two major contenders: string theory and loop quantum gravity. The main focus here in this thesis will be on loop quantum gravity.

In Chapter II, we will dive into some of the developments and roadblocks that led to Loop Quantum Gravity and then give a brief overview of some of its main concepts. Chapter III will explore Lorentz Invariance Violation in Loop Quantum Gravity. This violation is the crux of this thesis and so a few possible solutions will be more closely inspected in the following chapters. Chapter IV discusses the inclusion of matter in quantum gravity. Although physicists still do not have a good idea of exactly how matter comes about in quantum gravity, such an inclusion could arguably preserve

Lorentz invariance. Chapter V will introduce Deformed Special Relativity, which is one of the possible solution to Lorentz Violation. This form of special relativity has been tested via astronomical observations and although results did not look great, further considerations will still be given, which may lead to a possible solution.

CHAPTER II:

INTRODUCTION TO LOOP QUANTUM GRAVITY

One could argue that the history of quantum gravity begins all the way back to 1916 when Einstein realized that atoms also have gravitational energy and thus quantum mechanics would have to apply to gravity as well. A summary of the development of quantum gravity can be found by Ashtekar [9]. In quantizing gravity, one has to recognize that in non-relativistic quantum mechanics particles do not have a well-defined trajectory $x(t)$ and there is usually only a probability amplitude $\Psi(x, t)$. With this, there would appear to be a lack of a space-time geometry in quantum gravity, leading to an exclusion of causality, time, scattering states, and black holes in the formulation of quantum gravity. There were at least four approaches in attempting to include these common notions: a canonical approach, a covariant approach, perturbative string theory, and the anti-de Sitter/conformal field theory (AdS/CFT) conjecture.

The canonical approach tried to include causality by recognizing the Hamiltonian formulation of general relativity and using it as a way for quantization. Here, the idea of causality is recognized by how operators on the fixed three-manifold commute. The variable used here was the three-metric on a spatial segment. Einstein's equations can then be broken down into two groups: four constraints on the metric and its conjugate momentum, and six evolution equations [24]. From this, general relativity could be seen as the dynamical theory of three-geometries. Wheeler called this *geometrodynamics* [20]. The canonical approach would go on to attribute internal quantum numbers of elementary particles to non-trivial, microscopic topological configurations. In other words, particles are equivalent to and arose from topological layouts. Development in this approach, however, became stagnant when it was realized that other aspects of elementary particles played little to no roles here, such as quantum electrodynamics and the emergence of gravitons.

The covariant approach took a more opposite route, taking particles into consideration first [21][22][23]. The metric tensor $g_{\mu\nu}$ is now split into two parts:

$$g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{G}h_{\mu\nu} \tag{1}$$

where $\eta_{\mu\nu}$ is a background, kinematical metric, usually chosen to be flat, G is the gravitational constant, and $h_{\mu\nu}$ is a dynamical field representing the deviation of the physical metric from the chosen background. From this, it would only be $h_{\mu\nu}$ that is quantized and a discrete amount or quanta of $h_{\mu\nu}$ would propagate onto the background space-time with metric $\eta_{\mu\nu}$. Assuming the background to be flat, the gravitons can be shown to arise via the Casimir operators of the Poincaré group when the quanta of $h_{\mu\nu}$ has spin two and rest mass zero. This perturbation and expansion technique at first seemed promising; however, it was later found to be non-renormalizable [25]. It was however later realized from perturbative methods in electroweak interactions that this approach would be normalizable but only at low energies or large distances. This also would lead quantum gravity to differ from general relativity at high energies or at scales near the Planck length. The difference mainly comes from the simultaneous presence of quantum mechanical effects and curvature, both of which does not appear to be present at the same time in quantum mechanics or general relativity, but is presumed to be present in quantum gravity. Of course at large distances or low energies, gravity is normally too weak, making a theory of quantum gravity less necessary or at least difficult to test and verify.

The third approach to quantum gravity was string theory. In string theory, point particles are replaced by one dimensional extended objects called strings. Particle-like states are then associated with modes of excitations of the string. Without intention, string theory already had a mode of spin two, massless excitation, which translates to a graviton and thus would account for gravity at the quantum scale. Perturbation could be applied here since strings are assumed to be in flat space-time [9]. This would bring in new parameters, making the theory more non-local. Also, renormalization is not necessary here since string theorists believed that perturbation is finite to all orders. However, perturbative string theory does face the issues of ultraviolet finiteness and the lack of non-perturbative structures.

The last approach is the anti-de Sitter or conformal field theory (AdS/CFT) conjecture, which still relied on particle physics [9]. However, now it provides a relationship between string theory and quantum field theory by forming a non-perturbative form of string theory with certain boundary conditions. These boundary conditions exists on the AdS bulk spacetimes and would be where string theory is equivalent to certain gauge theories. Bulk here refers to a hyperspace or higher-dimensional space where a brane, which is an object that generalizes a point particle to higher dimensions, lives. At first the combination of string theory and quantum field theory would seem to resolve whatever problems that was more unique to either theory, but it still ran into some key problems though. For example, this conjecture has a negative cosmological constant while the observed cosmological constant is positive. It was also realized that the non-perturbative string theory here fails to describe much of the macroscopic world.

The roadblocks of these approaches brings us to loop quantum gravity (LQG), which was introduced by Carlo Rovelli and Lee Smolin. LQG is background independent, although there is a background manifold, and does not require the use of perturbation theory. The basis of LQG deals mostly with a quantized geometry.

Quantum Geometry

In LQG, there are at least two general approaches to quantizing gravity. The first is finding a quantum geometry. The second is simply quantizing various elements in space, such as quantizing matter. For a quantum geometry, there appears to exist a fundamental length and time: the Planck length $l_p = \sqrt{\hbar G/c^3}$ and Planck time $t_p = \sqrt{\hbar G/c^5}$ [5]. The Planck length and time being the smallest measurable distance and time would indicate that a quantum geometry exists in a discrete spectrum, with each discrete length and time being proportional to the Planck length and time. Since many classical variables exist on a continuous spectrum, this would lead to there being geometric operators that quantizes these variables, resulting in only a discrete set of possible values. As stated by [5], this result obviously has to be true for there to be a quantum Minkowski space describing a quantum gravitational field.

An existing fundamental length implies that the fabric of spacetime at some microscopic level is seen to be discrete. The discreteness of space comes from the assumption that at some point space cannot be divided any further and that any smaller division has no physical meaning or is not measurable. This limit is often times assumed to be the Planck length ($\sim 10^{-35}m$), but it does not necessarily have to be. The Planck length l_p is said to have been derived by Planck when he considered combining the fundamental constants of $\hbar$, Newton's gravitational constant G, and the speed of light c in such a way as to get a constant with the unit of length [10]:

$$l^2 = \frac{\hbar G}{c^3} \qquad \longrightarrow \qquad l_p = \sqrt{\frac{\hbar G}{c^3}}$$

Another way to derive the Planck length is shown in [29], where Heisenberg's uncertainty principle is used, along with the Schwarzchild radius R and rest mass M.

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

$$R = \frac{2GM}{c^2}$$

$$M = \frac{E}{c^2}$$

From the uncertainty principle, a precise measurement in position not only means that the uncertainty Δx is smaller than the precision L ($\Delta x \leq L$), but also that the uncertainty in momentum Δp becomes larger, leading to a large momentum p. This translates to a large energy E where we can get a relativistic limit of $v \to c$, making the rest mass M negligible. This in turn makes the energy as $E = pc$. The Schwarzchild radius is used here as a limit to how small L can get before a black hole is formed. Solving the uncertainty principle for p, where $\Delta x = L$, and setting $R = L$, we can combine the above three equations to get

$$L = \frac{2GM}{c^2} = \frac{2pcG}{c^4} = \frac{2\hbar G}{2Lc^3} \qquad \longrightarrow \qquad L^2 = \frac{\hbar G}{c^3}$$

$$\longrightarrow L = l_p = \sqrt{\frac{\hbar G}{c^3}}$$

Input the known values of $\hbar$, G, and c, we can see that l_p has an extremely small value on the order of 10^{-35} m. A summary of its history and development can be found by Hossenfelder [10]. This derivation of the Planck length may be a better way to see its significance. Seeing how the uncertainty principle and Schwarzchild radius is used here, it is easier to see that the Planck length is the most precise length or distance that is measurable and anything more precise than that would either lead to a black hole or a great uncertainty in momentum. It is usually assumed then that a scale well above this leads to a smooth spacetime. However, near the Planck scale it can be argued that spacetime is more discrete.

Do not be confused here. Even though spacetime is deemed to be discrete near the Planck scale, it can still be described as a differentiable manifold, where a metric structure is defined by the expectation values of a gravitational field operator [17]. Space being discrete results in there being points or nodes which are called loops, and functions of loops on the three-manifold would be taken to be quantum states. With each neighboring loop a network is formed, called a *spin network*. This can be seen in Figure 1 [29]. From here a *spin foam* is formed, comprising of the summation of various spin networks. A visualization of the loops and networks would give an outlook of a lattice structure, consisting of a minimal length, minimal area, and minimal volume.

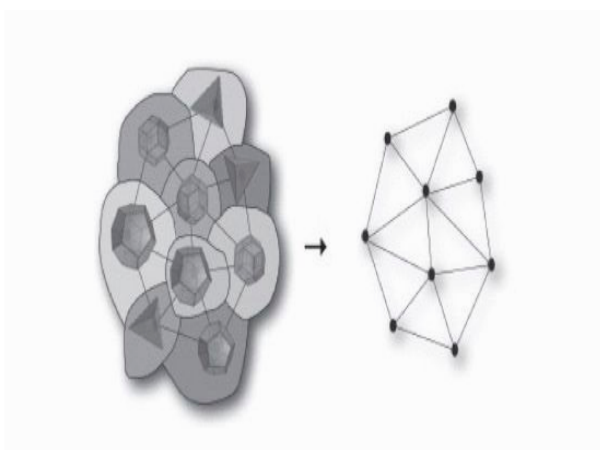

Figure 1: *A graph, where each quanta of space (left) translates to a node (right) [29]*

In order to have a better idea of what spin networks are, we need to take a closer look at some elements of general relativity. First off, in general relativity we have the covariant derivative:

$$\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\mu\lambda} A^\lambda \tag{2}$$

where $\Gamma^\nu_{\mu\lambda}$ allows us to connect neighboring points in spacetime; thus it is also called a *connection* or connection coefficient. In flat spacetime, this connection is zero and the covariant derivative is just the partial derivative. If the connection is torsion free (i.e., $\Gamma^\nu_{\mu\lambda} = \Gamma^\nu_{\lambda\mu}$) and if it is metric compatible (i.e., $\nabla_\sigma g_{\mu\nu} = 0$), then the connection can be determined entirely by the metric:

$$\Gamma^\nu_{\mu\lambda} = \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \tag{3}$$

in which case the connection is also called the Christoffel symbol. These connections comes about when considering parallel transport along a geodesic curve in curved spacetime, where we would have tangential and normal components. In LQG, the connection is determined a little bit differently. Here, we start by considering three-dimensional space instead of spacetime. We then have a set of three vector fields that are orthogonal: ${E_i}^a (i = 1, 2, 3)$. The inner product of these vectors results in

$$q^{ab} = {E_i}^a {E_j}^b \delta^{ij} \tag{4}$$

where we have flat space in coordinates i, j on the right-hand side and we have curved space with coordinates a, b on the left-hand side [30]. From this equation, we can see that the metric is formed from ${E_i}^a$, which are called "triads". Here, $a, b, c, ... = 1, 2, 3$ are spatial indices and $i, j, k, ... = 1, 2, 3$ are internal indices [17]. The internal indices represent a basis in the Lie algebra of $SU(2)$ or the three axis of a local triad. Seeing how the metric is formed from these triads that have internal indices, it is natural to ask "How does the covariant derivative apply to the triads?" For an object G^i, of only internal index, we can define a derivative similar to the covariant derivative as

$$\nabla_a G^i = \partial_a G^i + \Gamma^i_{aj} G^j \tag{5}$$

Here, the connection Γ^i_{aj} is called the *spin connection*, and similar to the connection in the covariant derivative, it would have to be defined externally. For a scalar (e.g., $G^i G_i$), in which this derivative would simply be a partial derivative, equation (5) would become

$$\nabla_a G_i = \partial_a G_i - \Gamma^j_{ai} G_i \tag{6}$$

For an object of mixed indices, the connections from both equations (5) and (6) would be used:

$$\nabla_a E^b_i = \partial_a E^b_i - \Gamma^j_{ai} E^b_j + \Gamma^b_{ac} E^c_i \tag{7}$$

Loop quantum gravity goes on to rely on these spin connections in forming the spin networks. To see how spin networks are actually formed, we would need to dive deeper into Yang-Mills theory, holonomies, loop representation, Asthekar variables, etc., which we will not do here, but can be found in [17], [29], and [30]. In short, LQG utilizes loop algebra and geometric operators to form the basic notions of length, area, and volume.

One of the approach that LQG takes in quantizing length is to analyze the situation in 2 + 1 gravity (two spatial dimension and one time dimension). Apparently, the main reason for this is because length operators are most easily analyzed here, and ignoring a cosmological constant, 2 + 1 gravity would be flat and so Lorentz symmetry would not be much of a concern [5]. Here it is predicted that for spacelike distances there would be a continuous spectrum, while for timelike distances there would be a discrete spectrum. The length operator $\hat{L}$ would act diagonally on spin networks whose edges are labeled by SO(2,1) and its spectrum would be

$$\hat{L}\Psi = l_p\sqrt{\rho^2 + \tfrac{1}{4}}\ \Psi \tag{8}$$

$$\hat{L}\Psi = i\tau_p\epsilon\sqrt{-n(n-1)}\ \Psi \tag{9}$$

where $\rho \in \mathbb{R}_+$ or ($\epsilon = \pm 1$, $n \in \mathbb{N}$) label the unitary representations of SO(2,1). Here Ψ is the quantum state of the gravitational field. There apparently also exists a quantization ambiguity resulting from the regularization procedure, leading to an alternative length spectrum [5]:

$$\hat{L}_s\Psi = l_p\rho\Psi \tag{10}$$

$$\hat{L}\Psi = i\tau_p\epsilon(n - \tfrac{1}{2})\Psi. \tag{11}$$

From here, we can begin to touch upon the question of how these length operators and their spectrum changes under a moving or boosted reference frame. There are at least two possibilities. The first is there may be a change in the operators for a boosted observer. This would lead to a change in the spectrum. The second is that the spectrum would not change but the operator and the quantum state would change [5]:

$$\hat{L}_{boosted}(\beta) = \mathrm{U}(\beta)\hat{L}U^{-1}(\beta) \tag{12}$$

$$\Psi_{boosted}(\beta) = \mathrm{U}(\beta)\Psi \tag{13}$$

$$\hat{L}_{boosted}\Psi_{boosted} = \mathrm{L}\Psi_{boosted} \tag{14}$$

where $\mathrm{U}(\beta)$ is the operator corresponding to a Lorentz boost in a representation of the Lorentz group.

From what has been shown, loop quantum gravity has at least two assumptions or postulates:

1. Gravity may be weak but at some small enough scale it becomes strong and unavoidable
2. At some small scale, space may become discrete, having a minimum length

The second postulate of a minimum length implies that Lorentz invariance is not preserved, especially in a boosted frame of reference where the minimum length would be seen to be contracted according to Lorentz contraction. It will be shown that these postulates have a lot of implications, some of which may be key to resolving Lorentz invariance violation. Lorentz invariance and its violation will be the next topic of exploration in Chapter III.

CHAPTER III:
LORENTZ VIOLATION

Lorentz invariance starts with the principle of relativity. This principle goes as far back as Galileo or even farther, and simply states that the laws of physics is the same in all applicable frames of reference (usually inertial). From here, it is easy to see that there is a symmetry that leads to experimental results being independent of a laboratory's orientation (rotational invariance) or its velocity (boost invariance) [31]. These invariances are what makes up the core of Lorentz invariance. They essentially are transformations that does not change experimental results. In special relativity, these transformations generally leave the spacetime interval (more specifically, the metric tensor) invariant and are usually represented in matrix notation. For example, we have the spacetime interval of $(\Delta s)^2$ in flat spacetime:

$$(\Delta s)^2 = \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu \tag{15}$$

where we have the Minkowski metric $\eta_{\mu\nu}$:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Spatial rotations or boosts are transformations that are described via a matrix $\Lambda^{\mu'}_{\nu}$:

$$x^{\mu'} = \Lambda^{\mu'}_{\nu} x^\nu. \tag{16}$$

If this transformation is Lorentz invariant, then when applied to the spacetime interval, we should get

$$(\Delta s)^2 = (\Delta x)^T \eta (\Delta x) = (\Delta x)^T \Lambda^T \eta \Lambda (\Delta x) \tag{17}$$

$$\rightarrow \qquad \eta = \Lambda^T \eta \Lambda$$

In index notation, this becomes

$$\eta_{\rho\sigma} = \Lambda^{\mu'}_{\rho} \eta_{\mu'\nu'} \Lambda^{\nu'}_{\sigma} = \Lambda^{\mu'}_{\rho} \Lambda^{\nu'}_{\sigma} \eta_{\mu'\nu'} \tag{18}$$

where the order does not matter. The matrices or transformations that satisfy equation (18) are deemed the Lorentz transformations. An example of a rotation in the x-y plane (where the rotation angle θ is periodic with period 2π) is [32]

$$\Lambda^{\mu'}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\theta & sin\theta & 0 \\ 0 & -sin\theta & cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

whereas a boost in the x-direction with boost parameter ϕ (defined from $-\infty$ to ∞) is

$$\Lambda^{\mu'}_{\nu} = \begin{pmatrix} cosh\phi & -sinh\phi & 0 & 0 \\ sinh\phi & cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

When applied to an object's coordinates, it becomes obvious that these transformations are simply a coordinate transformation. In fact, there are at least two types of Lorentz transformations: observer transformation and particle transformation [31]. An observer transformation is where new coordinates are chosen, such as doing a translation where the origin would be moved or switching to polar coordinates. A particle transformation is where the system itself is changed, either boosted or rotated. Either transformations would still result in a coordinate transformation.

At this point, it becomes relevant to ask "How does loop quantum gravity or quantum gravity in general violate Lorentz invariance?" The violation begins with the discreteness of the geometry, which gives rise to a lattice structure where there is a minimum area and minimum volume. This violation is more apparent when considering a boosted reference frame. Any measured length in an inertial reference frame will be observed as contracted from the perspective of the boosted reference frame, especially one that is moving near the speed of light. This means that a measured Planck length would be contracted; therefore, the Planck length would no longer be a constant, leading to a continuous spectrum [5]. Of course, applying Lorentz symmetry that was formulated in a flat Minkowski spacetime to a quantum geometry could simply be wrong; however we are assuming that such symmetry is applicable since there are no obvious reasons or signs as to why it would not be.

Another problem with the violation of Lorentz invariance is that it becomes more pronounced when considering the interaction of quantum fields. Gambini and Pullin

[3] argue that to avoid these large violations, one needs to consider non-local interactions in the quantum fields similar to those in string theory. They first considered the type of matter involved and saw that the background quantum states are peaked around a definite value of the ADM mass when considering the solutions to spherically symmetric quantum space-times. Their treatment of spherically symmetric quantum space-times led to the result of the matter fields looking like discrete versions of the continuum equations, where this discreteness arises from the background quantum geometry. The matter field is seen to exist at the vertices of the spin network of the background quantum state and is thus described by $\vec{\phi}$, a vector of values representing the values of the field at the vertices of the spin network. From this, the quantum state for the gravitational and matter field is $|\tilde{g}, \vec{k}, M, \vec{\phi}\rangle$, where $\tilde{g}$ is the equivalence class of graphs under diffeomorphisms of g, $\vec{k}$ is a vector of valences, and M is the value of the ADM mass. To merge this with quantum field theory on a classical space-time, the components of $\vec{\phi}$ with values of the fields at particular coordinates $\phi(r)$ needs to be identified and would depend on the quantum state of matter and gravity [3]. If this quantum state is in a superposition of values of the mass and valences $\vec{k}$, then this would imply that each value of $\phi(r)$ will correspond to a superposition of the components of $\vec{\phi}$. The need for non-local interactions then comes when considering the calculation for the quantity of self-energy from Collins *et al.* [1]. Here they showed that the calculations on a discrete quantum space-time results in dispersion relations that are of a lattice theory and not Lorentz invariant. Gambini *et al.* even shows that the second derivatives of the self-energy does not cancel, leading to large violations of Lorentz invariance [4].

One of the ways to consider non-local interactions was to consider a $\lambda\phi^4/4!$ theory, where the interaction in momentum space is replaced by $\lambda[exp(-\alpha^2(p_0^2 - \vec{p}^2)^2)\phi(p_0, \vec{p})]^4/4!$ and α is a function of ΔM [3]. In general, this approach has two requirements: the interaction should be Lorentz invariant and the exponential nature of the factor is to make it compatible with the non-locality. Interestingly, the non-local exponential types of interactions considered here have been studied in the context of string theories and is also the first time where loop quantum gravity limits the type of matter that can be used.

The preservation of Lorentz invariance in quantum gravity is apparently very difficult, if not impossible. Such difficulty has led many theorists to simply assume that Lorentz invariance is violated, and it may be better or easier to constrain this violation so as to better see when it occurs. Limiting this violation can be done in many ways. One way is to use high-energy photons or particles. To better see why high-energy particles are used, consider that Lorentz violation is expected to occur near the Planck scale. This means that we would need a particle whose energy is close to that of the Planck mass $m_p \equiv \sqrt{\hbar c/G_N} \simeq 1.22 \times 10^{19}\ GeV/c^2$. Such energy has yet to be seen by any Earth based instruments and even the most energetic particles that have been detected only have energies of $E \lesssim 10^{11}\ GeV/c^2 \sim 10^{-8}\ m_p$ [33]. In many quantum gravity models, Lorentz violation can be seen through modified dispersion relations for particles, generally of the form

$$E^2 = (pc)^2 + (mc^2)^2 + f(E, p; \mu; M) \tag{19}$$

where E and p are the particle energy and momentum, respectively; μ is a particle mass-scale, and M denotes the relevant quantum gravity scale (usually of the order of the Planck mass) [33]. The particles most often used as constraints are photons and electrons, and so quantum electrodynamics (QED) plays a significant role here as well. Thus, for electrons and photons, equation (19) respectively becomes

$$E_e^2 = m_e^2 + p^2 + f_e^{(1)} p + f_e^{(2)} p^2 \tag{20}$$

$$E_\gamma^2 = (1 + f_\gamma^{(2)}) p^2 \tag{21}$$

where we have set $c = 1$ and expanded the function $f(E, p; \mu; M)$ in powers of the momentum and considered only the lowest order terms (p, p^2, p^3). Here, $f_e^{(1)}, f_e^{(2)}$, and $f_\gamma^{(2)}$ depend on the helicity state of the particles [33]. So far, equations (20) and (21) only applies for that of particles and not fields. To see what the modified dispersion relations are for fields, we would have to turn to Effective Field Theory (EFT) where there are Lorentz violation operators [34]. In [35], we can see that for fields the lowest order non-renormalizable operators acting on the QED Lagrangian leads to the modified dispersion relations of

$$\omega_\pm^2 = k^2 + \frac{\xi_\pm^{(n)} k^n}{M^{n-2}} \tag{22}$$

$$E_\pm^2 = p^2 + m_e^2 + \frac{\eta_\pm^{(n)} p^n}{M^{n-2}} \tag{23}$$

where equation (22) applies to photons while equation (23) applies to electrons, or more generally fermions. The constants $\xi_\pm^{(n)}$ and $\eta_\pm^{(n)}$ indicate the strength of the violation, taking on whole real numbers [34]. The plus and minus signs in equation (22) indicates right and left polarization respectively, while in equation (23) they indicate opposite helicity states. The values of n indicate the mass dimension of the Lorentz violation operators. A deeper dive into how these equations are derived and what they mean can be found in [33], [34], and [35]. In summary, the importance of equations (20), (21), (22), and (23) is that they are modified dispersion relations which leads to a constraint procedure allowing us to see where and when Lorentz invariance violation may effectively occur. For example, the Greisen-Zatsepin-Kuzmin (GZK) limit at $5 \times 10^{19} eV/c^2$ has been used as a cutoff for when Lorentz violation occurs, especially in explaining how heavier elements in ultra-high energy cosmic rays do not lose energy as they travel towards Earth [34, 35]. This means that if these equations are true, then we may not need to observe particles with energies anywhere near 10^{11} GeV/c^2 in order to see Lorentz violation; and if we cannot reach the limit where Lorentz violation effectively begins, then it is highly unlikely that Lorentz violation will ever be observed if it is such a thing to begin with.

The use of photons and fermions as constraints is a starting point in seeing how we can preserve Lorentz invariance. This approach leads into the consideration of matter in Loop Quantum Gravity, which is another way in how Lorentz invariance may be preserved. This will be explored more in the next chapter. So far, the constraints considered here assumes Lorentz violation not only exists but occurs at a certain limit. Other approaches does not make this assumption and instead tries to modify relativity so as to preserve Lorentz invariance in Loop Quantum Gravity. One such approach is that of Deformed Special Relativity, also known as Doubly Special Relativity, and is the main focus of Chapter V.

CHAPTER IV:

MATTER IN LOOP QUANTUM GRAVITY

The topic of matter in loop quantum gravity has now become more relevant. Not only is it needed to observe gravity at the quantum scale, but it would also be used in tests of Lorentz violation. So far there has yet to be a clear cut way of seeing how matter arises in quantum gravity. One approach in seeing how matter couples to quantum gravity is via scalar fields. This is because matter fields are often times seen as a form of a scalar field.

As mentioned above, quantum states exists at the nodes or loops of the spin network or graphs and are a result of loop functions. This means the background space-time geometry would be a kinematical Hilbert space and the matter fields can be scalars (such as $\vec{\phi}$), spinors, and 1-form gauge fields [2]. Scalar fields have been used as a "clock". This can be the case when you consider the equation of motion of a massless, free scalar field ϕ as

$$\partial_\mu(\sqrt{-det(g)}g^{\mu\nu}\partial_\nu\phi) = 0 \tag{24}$$

In a spatially homogeneous space-time, this reduces to

$$\partial_t(\sqrt{det(q(t))}N^{-1}\partial_t\phi) = 0 \tag{25}$$

The scalar field is then seen as a "clock" if the lapse function N is $\sqrt{det(q(t))}$. This means $\phi \propto t$, suggesting that we can use a massless scalar field as an internal clock, instead of the coordinate time t, to describe the dynamics, avoiding the "problem of time" in quantum gravity [2].

To see a scalar field with mass, one needs to consider the Higgs scalars. Here the background space-time is Minkowski space-time and the Hilbert space is usually constructed via Gaussian measure, leading to the standard Fock Hilbert space. A problem arises from the fact that the Hilbert space in loop quantum gravity is background independent while Gaussian measure is not diffeomorphism invariant. Therefore, considering the Higgs scalars in loop quantum gravity becomes problematic. A possible way to avoid this problem is summarized by [2] as using a set of bounded variables for quantizing the Higgs scalars. So far, this applies to compact scalar fields. Other scalar fields to be considered are non-compact scalars, parametrized scalars, etc., and even fermions.

In LQG, this begins by considering the Lagrangian for a scalar field [30]:

$$L = -\frac{1}{2}\int \Big[\partial_\mu \phi \partial^\mu \phi + V(\phi)\Big] d^3x \tag{26}$$

where $V(\phi)$ is a potential that indicates how the field interacts with itself. If the field has mass, then the potential would contain a term like $m^2\phi^2/2$ where m is the mass of the field. If the field is massless, then $V(\phi) = 0$ and the wave motion of the field would move at the speed of light. This wave motion is described through the equation of motion of the Lagrangian and is a wave equation [30]:

$$\partial_\mu \partial^\mu \phi - \frac{dV(\phi)}{d\phi} = 0 \tag{27}$$

Equation (27) basically comes from inputting equation (26) into the Lagrange equation of motion:

$$\frac{\partial L}{\partial \vec{r}_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{r}}_k} = 0$$

where $\vec{r}$ is the position vector and $k = 1, 2, 3$ indicates the position. The Hamiltonian then becomes

$$H = \frac{1}{2}\int \Big[\ \pi^2 + (\nabla\phi)^2 + (m\phi)^2\Big]\ d^3x \tag{28}$$

In equation (28), π is the canonical momentum conjugate to the field and is defined as $\pi = \delta L/\delta\dot{\phi} = \dot{\phi}$. Quantization of the Hamiltonian results in ϕ and π becoming operators ($\phi \rightarrow \hat{\phi}$, $\pi \rightarrow \hat{\pi}$) and their Poisson brackets become commutators [30]:

$$[\hat{\phi}(\vec{x}), \hat{\pi}(\vec{y})] = i\hbar\delta^3(\vec{x} - \vec{y}) \tag{29}$$

$$[\hat{\phi}(\vec{x}), \hat{\phi}(\vec{y})] = [\hat{\pi}(\vec{x}), \hat{\pi}(\vec{y})] = 0 \tag{30}$$

Expanding the field $\hat{\phi}$ in Fourier space gets us the relation between momentum and position space:

$$\hat{\phi}(\vec{x}, t) = \int \frac{e^{i\vec{p}\cdot\vec{x}}\hat{\phi}(\vec{p}, t)}{\sqrt{(2\pi)^3}} d^3p \tag{31}$$

From this, the Klein-Gordon equation, which describes the scalar field, becomes

$$(\frac{\partial^2}{\partial t^2} + \vec{p}^{\,2} + m^2)\phi(\vec{p}, t) = 0 \tag{32}$$

where we have set $c = 1$ [30]. This also applies to a harmonic oscillator of frequency $\omega(\vec{p}) = \sqrt{\vec{p}^{\,2} + m^2}$. Like the harmonic oscillator, the field and momentum operators would have creation and annihilation operators as well

$$\hat{\phi}(\vec{p}) = \frac{\hat{a}_{\vec{p}} + \hat{a}^{\dagger}_{-\vec{p}}}{\sqrt{2\omega(\vec{p})}} \tag{33}$$

$$\hat{\pi}(\vec{p}) = -i\sqrt{\frac{\omega(\vec{p})}{2}}(\hat{a}_{\vec{p}} - \hat{a}^{\dagger}_{-\vec{p}}) \tag{34}$$

with the creation and annihilation operators commuting as

$$[\hat{a}_{\vec{p}}, \hat{a}^{\dagger}_{-\vec{p}}] = \delta^3(\vec{p} - (-\vec{p})) \tag{35}$$

This allows us to quantize the Hamiltonian in equation (26) as [30]

$$\hat{H} = \int \omega(\vec{p}) \left\{ \hat{a}^{\dagger}_{\vec{p}}\, \hat{a}_{\vec{p}} + \frac{1}{2}[\hat{a}_{\vec{p}}, \hat{a}^{\dagger}_{\vec{p}}] \right\} d^3p \tag{36}$$

The steps taken so far have allowed us to begin quantizing the scalar fields. How does matter or mass actually arise in the quantized scalar fields? As stated, there is no obvious answer to this; even though we can see in equations (24) and (34) that there is a way to include a mass term, we still do not know of how such terms comes about without adding it ad-hoc, arbitrarily or forcefully. One approach is to see how matter or gravity couples to the classical scalar fields and then go from there. In [30], this begins with the Lagrangian action,

$$S = \int L dt$$

which for a field becomes

$$S = \int \mathcal{L}(\phi^i, \partial_\mu \phi^i) d^4x \tag{37}$$

Here, $\mathcal{L}$ is a Lagrange density and is a function of the fields and their spacetime derivatives. Looking at equation (24), we can see that for a curved spacetime where there is dependence on the metric and its determinant, equation (35) becomes

$$S = \int \Big\{ -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) \Big\} \sqrt{-det(g)}\, d^4x \tag{38}$$

From this, we eventually get that the Hamiltonian in equation (26) becomes

$$H = \int N \Big\{ \frac{\tilde{\pi}^2}{\sqrt{det(q)}} + \sqrt{det(q)} \Big[q^{ab}\partial_a\phi\partial_b\phi + V(\phi) \Big] \Big\} d^3x + N^a \tilde{\pi} \partial_a \phi \tag{39}$$

where $\tilde{\pi}$ is still the canonical conjugate momentum $\tilde{\pi} = \partial L / \partial \dot{\phi}$, and q^{ab} being the positive definite spatial metric living on the three-dimensional manifold Σ and is related to the metric via

$$q_{ab} \equiv g_{ab} + n_a n_b$$

where n is a vector field perpendicular to Σ and $a, b = 1, 2, 3$. In equation (39), we also have the lapse N and the shift vector N^a, which can be viewed as a scalar and vector living in Σ, respectively. These can be defined in terms of the metric as [29]

$$N = \sqrt{-g^{00}}$$

$$N_a = g_{a0}$$

In equations (36) and (37), we have the factors $\sqrt{-det(g)}$ and $\sqrt{det(q)}$, respectively. Such factors come from the fact that we are integrating a scalar along a volume. Multiplying these factors by a scalar function we would get an object called a *scalar density*, and applying it to equation (4) would result in densitized triads [17]:

$$\tilde{E}_i^a = \sqrt{det(q)} E_i^a$$

$$\rightarrow \tilde{q}^{ab} = det(q) q^{ab} = \sqrt{det(q)} E_i^a \sqrt{det(q)} E_j^b \delta^{ij}$$

$$= \tilde{E}_i^a \tilde{E}_j^b \delta^{ij} = \tilde{E}_i^a \tilde{E}_i^b \tag{40}$$

With this, we can see that the Hamiltonian from equation (37) becomes,

$$H = \int \frac{N}{\sqrt{det(q)}} \Big\{ \tilde{\pi}^2 + det(q) q^{ab} \partial_a \phi \partial_b \phi + det(q) V(\phi) \Big\} d^3x + N^a \tilde{\pi} \partial_a \phi$$

$$= \int \frac{N}{\sqrt{det(q)}} \Big\{ \tilde{\pi}^2 + \tilde{E}_i^a \tilde{E}^{bi} \partial_a \phi \partial_b \phi + det(q) V(\phi) \Big\} d^3x + N^a \tilde{\pi} \partial_a \phi \tag{41}$$

We now have the Hamiltonian in terms of Ashtekar variables and it clearly has two parts. The first part with the integral is how the scalar field contributes to the Hamiltonian constraint. Integrating the second part of equation (39) (i.e., $N^a \tilde{\pi} \partial_a \phi$) with respect to the three-dimensional space is how the scalar field contributes to the diffeomorphism constraint. These contributions of the scalar field, when added to the Hamiltonian constraint and the diffeomorphism constraint, is how gravity couples to the scalar field [30].

So far we have seen how gravity can be coupled to the scalar field and how such fields can be quantized. Other approaches have used parameterized field theory and have mainly tried coupling fermions to gravity in order to see how matter through gravity arises [2, 16, 17]. Classically we know that matter goes with gravity, although here we do not know exactly how matter arises in quantum gravity. Therefore, in the context of this thesis, we can only make an argument as to how matter in loop quantum gravity can preserve Lorentz invariance. The argument is simply that the size of observable matter constrains Lorentz invariance to be preserved. For example, any length contraction that would not preserve the Planck length would not be of concern since the matter that would be used to make such an observation would have a size significantly larger than the Planck length itself. This is similar to the argument in the previous chapter, regarding the GZK limit.

A last approach in preserving Lorentz invariance at the Planck scale is to take a closer look at how special relativity or general relativity behaves at such scale. At the scale of the Planck length, it can be assumed that gravity is strong. This assumption of a strong gravity automatically results in a curved spacetime. This means that we need to investigate how lengths contract in a curved spacetime and formulate a transformation rule. This approach requires a closer look at how basis and unit vectors change in a curved space. The reason to focus on basis and unit vectors is because the Planck length can be considered to be the magnitude of a basis or unit vector, which forms all of space and does not change magnitude regardless of the frame of reference. This translates to a discrete spacetime where the motion of all objects is seen to be discrete and not continuous. This discreteness or minimum length applied to that of special relativity leads to some physical implications, which will be explored in the next chapter.

CHAPTER V:

DEFORMED SPECIAL RELATIVITY

So far there are two main attempts at resolving the issue of Lorentz violation in Loop Quantum Gravity: the inclusion of matter and deformed special relativity.

Also known as doubly special relativity, deformed special relativity (DSR) begins with two postulates [26]:

- The relativity principle holds even at the Planck scale.
- The Planck mass κ (or length where length $\lambda = \kappa^{-1}$) is observer independent, along with the speed of light c.

These postulates mean that if the Planck length is a fixed invariant minimal length, then it can be measured by any observer and the measured value should be the same for all inertial observer in any frame of reference. The development of DSR assumes this observation would be in flat Minkowski space and is realized via quantum deformation of the Poincare algebra of symmetries [6]. This quantum deformation of the Poincare algebra would require some mention of quantum algebra, particularly quantum groups.

Quantum groups here deal mostly with non-commutative algebra (i.e. Hopf algebra). In the case of DSR with its two postulates and flat Minkowski space, we would have the Poincaré group, where there are not only ten dimensional groups of symmetries, corresponding to rotations, boosts, and translations, but there is also a second scale κ. Because of this scale, the De Broglie dispersion relation $E^2 = c^2p^2 + c^4m^2$ would have to be modified. Although [26] did not mention any specific modifications, [13] did suggest that the modification may depend on the measurement procedure. For example, a modification of the dispersion relation may result in the form of $E^2 - c^2p^2 + f(E, p; \kappa^{-1}) = 0$, where the function $f(E, p; \kappa^{-1})$ is the same for all inertial observer and all inertial observers would agree on the leading κ^{-1} dependence of $f : f(E, p; \kappa^{-1}) \simeq \kappa^{-1} c p^2 E$. This scale κ taken in consideration with other constants in quantum gravity (i.e. $c, G, \hbar$, and Λ) may say that in the limit of quantum gravity, we do not have Minkowski spacetime [26]:

1. $\lim_{G,\Lambda \to 0} \sqrt{\frac{G}{\Lambda}} = \kappa^{-1} \neq 0$
2. $\lim_{G,\hbar \to 0} \sqrt{\frac{\hbar}{G}} = \kappa \neq 0$

It is unclear if either of these limits are true. In the first case where we have a positive cosmological constant Λ, the excitations of a three dimensional quantum gravity would transform under representations of the quantum deformed deSitter algebra $SO_q(3,1)$, which is seen to be a generalization of Lie algebra [27]. Here, q is the deformation parameter and is related to Λ by

$$q = e^{i\sqrt{\Lambda}\hbar G} = e^{i\pi/r} \tag{42}$$

Being a generalization of Lie algebra, the quantum algebra would have the following commutation relations, which are still assumed to be antisymmetric and obeying the Jacobi identity,

$$
\begin{aligned}
[M_{2,3}, M_{1,3}] &= \frac{1}{z} sinh(zM_{1,2}) cosh(zM_{0,3}) \\
[M_{2,3}, M_{1,2}] &= M_{1,3} \\
[M_{2,3}, M_{0,3}] &= M_{0,2} \\
[M_{2,3}, M_{0,2}] &= \frac{1}{z} sinh(zM_{0,3}) cosh(zM_{1,2}) \\
[M_{1,3}, M_{1,2}] &= -M_{2,3} \\
[M_{1,3}, M_{0,3}] &= M_{0,1} \\
[M_{1,3}, M_{0,1}] &= \frac{1}{z} sinh(zM_{0,3}) cosh(zM_{1,2})
\end{aligned}
$$

$$[M_{1,2}, M_{0,2}] = -M_{0,1}$$
$$[M_{1,2}, M_{0,1}] = M_{0,2}$$
$$[M_{0,3}, M_{0,2}] = M_{2,3}$$
$$[M_{0,3}, M_{0,1}] = M_{1,3}$$
$$[M_{0,2}, M_{0,1}] = \frac{1}{z} sinh(zM_{1,2}) cosh(zM_{0,3})$$

where M_{ij} $(i, j = 0, 1, 2, 3)$ is the generator of the Lorentz transformation and the parameter z is related to q by $z = lnq$. In the limit that $z \to 0$, we get back out the standard SO(3,1) algebra. Notice here that the commutations do not result in linear function generators but instead results in analytic functions of them. The generators can be found via

$$M_{ij} = [J_i, J_j] = i\epsilon_{ijk} J_k \tag{43}$$

where J_1, J_2, and J_3 are the generators of the rotation around the x, y, and z-axis, respectively [28]. In matrix notation, they have the forms of

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

So far, we can see that SO_q(3,1) is a deformation of deSitter algebra. Similarly, there is also a deformation of Poincaré algebra, called κ-Poincaré algebra. To see this, we can rescale some of the generators [26]:

$$E = \sqrt{\Lambda}\hbar M_{0,3}$$
$$P_i = \sqrt{\Lambda}\hbar M_{0,i}$$
$$M = M_{1,2}$$
$$N_i = M_{i,3}$$

If G or Λ is small, then we have $z = lnq = i\sqrt{\Lambda}\hbar G \approx \sqrt{\Lambda}\hbar\kappa^{-1}$ where $G = \kappa^{-1}$. With this, the commutation relations would obviously change. For example, $[M_{2,3}, M_{1,3}]$ and $[M_{0,2}, M_{0,1}]$ becomes

$$\begin{aligned}[M_{2,3}, M_{1,3}] &= \frac{1}{z} sinh(zM_{1,2})cosh(zM_{0,3}) \\ &= \frac{\kappa}{\hbar\sqrt{\Lambda}} sinh\Big(\frac{\hbar\sqrt{\Lambda}M}{\kappa}\Big)cosh\Big(\frac{E}{\kappa}\Big) \\ &= [N_2, N_1] \end{aligned} \tag{44}$$

$$\begin{aligned}[M_{0,2}, M_{0,1}] &= \frac{1}{z} sinh(zM_{1,2})cosh(zM_{0,3}) \\ &= \sqrt{\Lambda}\hbar\kappa sinh\Big(\frac{\hbar\sqrt{\Lambda}M}{\kappa}\Big)cosh\Big(\frac{E}{\kappa}\Big) \\ &= [P_2, P_1] \end{aligned} \tag{45}$$

In the limit of $\Lambda \to 0$ and κ being constant, equations (44) and (45), along with the rescaled generators, can be generalized to become

$$[N_i, N_j] = -M\epsilon_{ij} cosh(E/\kappa)$$
$$[M, N_i] = \epsilon_{ij}N^j$$

$$
\begin{aligned}
&[N_i, E] = P_i \\
&[N_i, P_j] = \delta_{ij}\kappa sinh(E/\kappa) \\
&[M, P_i] = \epsilon_{ij}P^j \\
&[E, P_i] = 0 \\
&[P_2, P_1] = 0
\end{aligned}
$$

These generalized commutations are what is called the three dimensional κ-Poincaré algebra in the standard basis and are a deformation of the Poincaré algebra. A more exact derivation of these relations can be found in [36]. As a result, κ-Poincaré algebra is also a quantum algebra and we can change the basis of the generators arbitrarily as to get back out the classical Lorentz transformation generators. This has been shown in [37] and such a basis is called a bicrossproduct. In this basis, the commutation relations become

$$
\begin{aligned}
&[N_i, N_j] = -\epsilon_{ij}M \\
&[M, N_i] = \epsilon_{ij}N^j \\
&[N_i, E] = P_i \\
&[N_i, P_j] = \delta_{ij}\frac{\kappa}{2}\left(1 - e^{2E/\kappa} + \frac{\vec{P}^2}{\kappa^2}\right) - \frac{1}{\kappa}P_iP_j \\
&[M, P_i] = \epsilon_{ij}P^j \\
&[E, P_i] = 0 \\
&[P_1, P_2] = 0
\end{aligned}
$$

The DSR model in this basis, following these relations is called DSR1. Getting back out the classical Lorentz transformation generators also means that this model is in flat spacetime, while at the same time containing an observer independent scale κ. This makes DSR1 appear to be of appropriate use in loop quantum gravity since it would not only preserve the speed of light but also the Planck length. However, depending on the dispersion relation, the speed of massless particles, such as photons, can begin to vary. Assuming that the relation $v = dE/dp$ still holds and depending on the sign of $f(E, p; \kappa^{-1})$, it is found that the speed of massless particles in the low energy limit may begin to decrease as energy decreases, although the speed of light ($\approx 3 \times 10^8 m/s$) is still the maximum observable speed [39]. For example, in [39] it is found via the modified dispersion relation of

$$E^2 = \vec{p}^2 + m^2 + \lambda E \vec{p}^2$$

that the velocity of particles become

$$v \simeq 1 - \frac{m^2}{2E^2} + \lambda E$$

where λ is the wavelength and is deemed to be positive. Some calculations suggest that this energy-dependent speed holds only for seeing particles as waves, while for point particles, the speed of light is expected to not be energy-dependent. See [26] for more details.

There are other models of DSR that are formulated slightly differently but can still have similar results. We will not go into too much details but an example here is in [38], where similar to $E^2 - c^2p^2 + f(E, p; \kappa^{-1}) = 0$, the dispersion relations used is $E^2 = p^2 + m^2 + \lambda E^3 + \ldots$ (in natural units). In terms of frequency, this becomes $E^2 {f_1}^2(E; \lambda) - p^2 {f_2}^2(E; \lambda) = m^2$. Here, λ is instead a proportionality factor of the order of the Planck length and may be positive or negative. This model is sometimes called DSR2. Unlike DSR1, it has no deformations but depending on how one defines the functions f_1 and f_2, it can be seen that the dispersion relations of DSR2 is still the same or at least of the same leading-order modification as those in the κ-Poincaré group [38]. Like DSR1, it still results in an energy-dependent speed of light for massless particles in the low energy limit. For example, if $f_1 \neq f_2$ then from dE/dp the speed of light is [38][41]

$$c = \frac{dE}{dp} = \frac{f_3}{1 - \frac{Ef_3'}{f_3}}$$

where $f_3 = f_2/f_1$ and $f_3' = df_3/dE$. This of course shows that the speed of light is not constant but becomes energy-dependent.

Basically, DSR not only has the maximum speed of light as a constant but also the Planck length as a constant as well. If this is the case, then does that mean there is a factor, like the Lorentz factor, at play here that keeps the Planck length constant, similar to how the Lorentz factor keeps the speed of light constant and prevents speeds faster than the speed of light from existing? Such a question has been asked, and like in DSR1 and DSR2 it has been found that the speed of massless particles would not be constant but would vary depending on frequency or the energy of the particle [5]. It has been found that c is only the speed of light at low energies, $E \to 0$, and approaches ∞ as E reaches the Planck energy [7]. This varying speed of light may preserve Lorentz invariance; however, it has not been observed yet, with some evidence even contradicting it. For example, in 2009 the Fermi Gamma-ray Space Telescope made an observation from a burst and saw that a 31 GeV photon arrived at approximately the same time as other photons of the same burst [8]. This obviously supports the classical notion that the speed of light for massless particles is not frequency or energy dependent.

An example of the effects of this varying speed of light is how it affects length. Seeing as the speed of light is used to define unit length, we see that a theoretical distance is given, with $T_{(E)}$ as the measured time of travel, as [5]

$$d = \frac{T_{(E)}}{2} \times c(E), \tag{46}$$

which is different from an effective distance of

$$d_{eff} = \frac{T_{(E)}}{2} \times c\,. \tag{47}$$

As stated above, the speed $c(E)$ would increase with energy, and if d is to remain constant, then $T_{(E)}$ would decrease. This decreasing $T_{(E)}$ leads to the measured distance d_{eff} decreasing, even though it theoretically should remain constant. Therefore, the measurement of a single distance in DSR has no meaning. Distances in DSR only begin to have meaning when considering their ratios, since their ratios does not depend on $c(E)$ [5].

Relative Locality

We have now begun to see that in DSR the Planck length and the maximum speed of light can be preserved to remain constant, all the while resulting in a varying speed for massless particles which normally travels at the speed of light. For the rest of this thesis, we will take a closer look at the postulates of loop quantum gravity and see how the Lorentz transformations or the Lorentz factor is affected.

The first postulate of a strong gravitational field in quantum gravity translates to significant curvature in spacetime. If we are working near the Planck scale with strong curvature, then the assumption of local flatness would not apply (or would be much more limited) since local flatness would require one of two things: 1) shrink our frame of reference to be significantly smaller than the Planck length, or 2) expand our frame of reference to be much larger than the Planck scale, both of which would not only make local flatness viable but also make gravity significantly weaker, allowing gravity to be ignored all together. We do not want this; thus, we are not assuming local flatness at all, and if we do, then we would need to define a limit at which it is applicable. This means that we have to take curvature into account when considering the Lorentz invariance and Lorentz transformations.

Let us start by considering the notion of relative locality. In this framework, there is no such thing as absolute locality, resulting in different observers seeing different spacetimes and the spacetimes are energy and momentum dependent. From this, we can define the *Principle of Relative Locality* [40]:

Physics takes place in phase space and there is no invariant global projection that gives a description of processes in spacetime. From their measurements local observers can construct descriptions of particles moving and interacting in a spacetime, but different observers construct different spacetimes, which are observer-dependent slices of phase space.

The usage of phase space in this principle comes from using the Planck mass m_p as an energy scale. Although in [40] it is assumed that gravitational effects (such as curvature distorting spacetime) may be ignored, we will not be assuming that here, as stated above. Instead, we will be utilizing the basic idea of relative locality where different observers in different local reference frames observes different spacetimes and seeing how that might preserve the Planck length.

Deriving the Lorentz transformations in curved spacetime would also require deriving the Lorentz factor in curved spacetime. A quick way to derive the Lorentz factor is to consider an object moving in the x direction with velocity v. An observer moving in the x' frame of reference would normally see the object with the same velocity; however, if the object is moving through space with non-uniform curvature, then in the x' frame the object may instead be observed to have velocity v', as seen in the figures below. Both velocities are less than and proportional to the speed of light. The curvature of space may also change the value of the speed of light by a proportionality χ. The justification for this possibility is that a curved space would also change the metric, thus changing the speed of light in that curved frame to be different than the speed in flat space. Of course accounting for the curvature may keep the speed of light the same in every frame of reference.

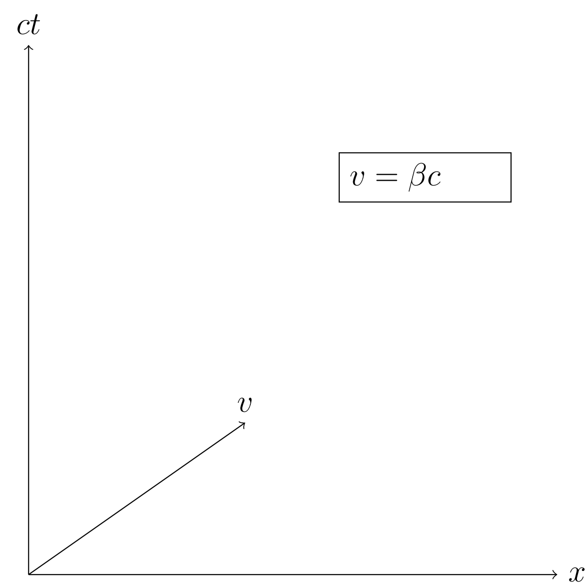

Figure 2: *An object moving in the x frame with velocity v*

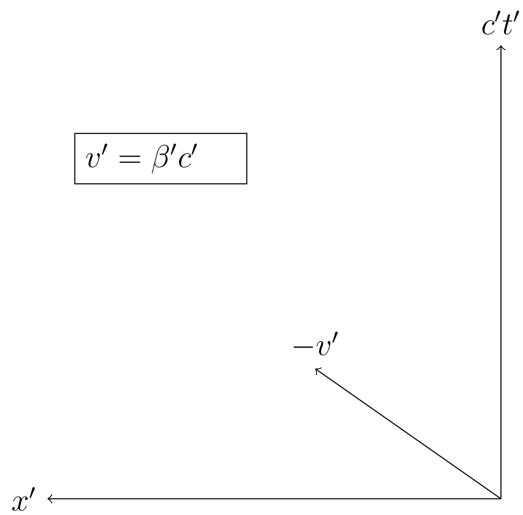

Figure 3: *The same moving object from the boosted x' frame*

$$v = \beta c \tag{48}$$

$$v' = \beta' c' = \frac{\beta'}{\chi} c \tag{49}$$

$$c = \chi c' \tag{50}$$

Using the relation of

$$c = \frac{l_p}{t_p} \tag{51}$$

we can see that in the x' frame, c' from equation (50) is equal to

$$c' = \frac{l'_p}{t'_p} \tag{52}$$

where

$$l'_p = \frac{l_p}{k_l}$$

$$t'_p = \frac{t_p}{k_p}$$

To clarify, the Planck length l_p, Planck time t_p, and speed of light c in the x frame of reference is cautiously assumed to be proportional to the Planck length l'_p, Planck time t'_p, and speed of light c' in the x' frame of reference. This is where the proportionality factors k_l, k_p, and χ comes from. Their exact form or values will not be explored here but could be a result of curvature or the metric being different in different frames of reference. Substitute this into equation (50) and we see that

$$\frac{l_p}{t_p} = \frac{\chi l'_p}{t'_p} = \frac{\chi k_p l_p}{k_l t_p}$$

$$\Rightarrow \frac{\chi k_p}{k_l} = 1$$

$$\chi = \frac{k_l}{k_p} \tag{53}$$

Equation (53) is an obvious result, telling us that the proportionality between the speed of light in different reference frames comes from the ratio between the propotionalities of their Planck length and Planck time. In flat spacetime or spacetime with uniform curvature, $\chi = 1$ or $k_l = k_p$, which leads to $c = c'$. This could still be the case regardless of whether or not there is uniform curvature, which we will consider in our derivation; however, it may be safer to assume that $k_l \neq k_p$ in this scenario.

The distance traveled by the object is then seen in the x and x' frame respectively as

$$x = \gamma'(x' + v't') = k_x l_p \tag{54}$$

$$x' = \gamma(x - vt) = k'_x l'_p = \frac{k'_x l_p}{k_l} \tag{55}$$

both of which are proportional to the minimum length l_p. Also in equations (54) and (55) we have t and t' as

$$t = k_t t_p \tag{56}$$

$$t' = k'_t t'_p = \frac{k'_t t_p}{k_p} \tag{57}$$

Any proportionality that l_p has in the x' frame to the x frame can be absorbed into k'_x. To clarify, we could absorb k_l into k'_x, resulting in $x' = k'_x l_p$. However, we will see that this is not necessary when we solve for γ. The same can be said for that of k_p being absorbed into k'_t. We then have γ as being proportional to γ' since it may be safer to assume that the Lorentz factor is not uniform or constant everywhere but changes proportionally from one frame to another:

$$\gamma = \eta\gamma' \tag{58}$$

If the Lorentz factor is uniform or the same in all frames of reference and does not differ by any proportionality from frame to frame, then we simply have $\eta = 1$ in equation (58). Of course, if the Lorentz factor takes curvature into account, then as curvature changes from one frame to another the Lorentz factor will change without the need of η. The inclusion of η is due to the Lorentz factor possibly changing and being different in a different frame of reference.

Now to find the Lorentz factor γ, we can multiply equations (54) and (55) together and simplify to get

$$xx' = \frac{k_x k'_x {l_p}^2}{k_l} = \frac{\gamma^2 k_x k'_x}{\eta k_l}({l_p}^2 + \frac{\beta' k'_t k_l c t_p l_p}{\chi k'_x k_p} - \frac{\beta k_t c t_p l_p}{k_x} - \frac{\beta\beta' k_t k'_t k_l c^2 {t_p}^2}{\chi k_x k'_x k_p})$$

$$\Rightarrow {l_p}^2 = \frac{\gamma^2}{\eta}({l_p}^2 + \frac{\beta' k'_t k_l c t_p l_p}{\chi k'_x k_p} - \frac{\beta k_t c t_p l_p}{k_x} - \frac{\beta\beta' k_t k'_t k_l c^2 {t_p}^2}{\chi k_x k'_x k_p}) \tag{59}$$

Now divide equation (59) by ${l_p}^2$ and using the relation from equations (51) and (53), we get

$$1 = \frac{\gamma^2}{\eta}(1 + \frac{\beta' k'_t}{k'_x} - \frac{\beta k_t}{k_x} - \frac{\beta\beta' k_t k'_t}{k_x k'_x}) \tag{60}$$

Solve for γ,

$$\gamma^2 = \frac{\eta}{(1 + \frac{\beta' k'_t}{k'_x} - \frac{\beta k_t}{k_x} - \frac{\beta\beta' k_t k'_t}{k_x k'_x})}$$

$$\Rightarrow \boxed{\gamma = \sqrt{\frac{\eta}{(1 + \frac{\beta' k'_t}{k'_x} - \frac{\beta k_t}{k_x} - \frac{\beta\beta' k_t k'_t}{k_x k'_x})}}} \tag{61}$$

As seen in equation (61), γ does not depend on k_l and so it was not necessary to absorb it into k'_x. With this, the Lorentz factor now not only depends on the ratio between the velocity of the boosted frame and the speed of light (i.e., β, β'), but also on the proportionalities of distance and time to that of the Planck length and time (i.e., k_x, k'_x, k_t, k'_t). If $\eta = 1$ ($\gamma = \gamma'$) and the distance and time proportionalities are equal ($k_t = k_x, k'_t = k'_x$), then this simply becomes

$$\gamma = \sqrt{\frac{1}{(1 + \beta' - \beta - \beta\beta')}} \tag{62}$$

In flat spacetime, $\beta = \beta'$, which makes this become the usual Lorentz factor:

$$\gamma = \sqrt{\frac{1}{(1 - \beta^2)}}$$

This could effectively still be the case if gravity is weak enough, resulting in $\beta \approx \beta'$ and thus their difference would be very small ($\beta' - \beta \approx 0$). The result of equations (61) or (62) could preserve the Planck length depending on the values that k_x, k'_x, k_t, k'_t take and also on the sign of β and β'. We might not know exactly how to find these values at this point but considering that equation (61) can turn back into the usual Lorentz factor gives us a clue. First off, we know that $0 \leq \beta \leq 1$, $0 \leq \beta' \leq 1$, since they are just the proportionality factor between velocity and the speed of light, with velocity being less than c. Secondly, from equations (51) through (57) we can see that the average velocities in the x and x' frame gives us

$$\overline{v} = \frac{k_x l_p}{k_t t_p} = \frac{k_x}{k_t} c$$

$$\overline{v'} = \frac{k'_x l'_p}{k'_t t'_p} = \frac{k'_x}{k'_t} c'$$

which indicates that $k_x < k_t$ and $k'_x < k'_t$. This also indicates that the value of their ratios in equation (61) is at least 1:

$$\left\|\frac{k_t}{k_x}\right\| \geq 1$$

$$\left\|\frac{k'_t}{k'_x}\right\| \geq 1$$

The ratios of these variables can also change the form of equation (61) in at least two ways:

1. If $\left\|\frac{k'_t}{k'_x}\right\| \gg \left\|\frac{k_t}{k_x}\right\|$, then

$$\gamma \approx \sqrt{\frac{\eta}{1+\frac{k'_t}{k'_x}\beta' - \frac{k'_t k_t}{k'_x k_x}\beta'\beta}} = \sqrt{\frac{\eta}{1+\frac{k'_t}{k'_x}\beta'(1-\frac{k_t}{k_x}\beta)}} \tag{63}$$

and if $\frac{k'_t}{k'_x}\beta' \gg \frac{k'_t k_t}{k'_x k_x}\beta'\beta$ (or $1-\frac{k_t}{k_x}\beta \approx 1$) then we have

$$\gamma \sim \sqrt{\frac{\eta}{1+\frac{k'_t}{k'_x}\beta'}}$$

2. If $\left\|\frac{k'_t}{k'_x}\right\| \ll \left\|\frac{k_t}{k_x}\right\|$, then

$$\gamma \approx \sqrt{\frac{\eta}{1-\frac{k_t}{k_x}\beta - \frac{k'_t k_t}{k'_x k_x}\beta'\beta}} = \sqrt{\frac{\eta}{1-\frac{k_t}{k_x}\beta(1+\frac{k'_t}{k'_x}\beta')}} \tag{64}$$

and if $\frac{k_t}{k_x}\beta \gg \frac{k'_t k_t}{k'_x k_x}\beta'\beta$ (or $1+\frac{k'_t}{k'_x}\beta' \approx 1$) then we have

$$\gamma \sim \sqrt{\frac{\eta}{1-\frac{k_t}{k_x}\beta}}$$

In both cases, length contraction may actually become a dilation depending on the sign of $k'_t\beta'/k'_x$ and $k_t\beta/k_x$, which could depend on the frame of reference or the curvature within those frames. This means that the Planck length in one frame of reference could instead look dilated in a boosted frame. The second conditions in both cases where $1 - \frac{k_t}{k_x}\beta \approx 1$ and $1 + \frac{k'_t}{k'_x}\beta' \approx 1$ seems to indicate that this condition only applies at low velocities where $\beta, \beta' \to 0$ since the lowest value that k'_t/k'_x and k_t/k_x can have is 1. At higher velocities, this second condition appears to no longer apply, and instead we would have to rely on equations (63) and (64).

Observability

Now as stated, equation (61) holds under a modified definition of Lorentz invariance where curvature is present, especially considering our postulates. However, under the insistence of inertial reference frames, where there is no curvature, we can still try to resolve Lorentz invariance violation under the postulate of discrete spacetime. Not only does this postulate arise out of the definition of the Planck length and Planck time being the minimum observable length and time respectively, but it also goes on to have some important implications on what is observable and what is not. The first implication is that any distance or time below that of the Planck length and Planck time can exist but would not be observable. From this we would see that any distance or time that is not a discrete size of the Planck length or Planck time would also not be observable, even if they are larger than the Planck length or Planck time. Take for example a distance of 3.1415 Planck length; it is greater than a Planck length but would be observed as 3 Planck lengths since 0.1415 Planck lengths is not observable. The second implication is that velocities slower than the speed of light would effectively be an average of total distance traveled over total time. By this we mean that continuous velocity and discrete velocity would be observed to effectively be the same. Take for example, a particle at the Planck scale moves a distance of one Planck length over ten Planck times, which results in a velocity of $0.1c$. Now the particle could be moving continuously during the total time of ten Planck times, but each distance per unit time would be smaller than a Planck length and thus is not observable until the total distance is at least a Planck length. This is effectively the same as the particle being stationary for nine Planck times and then moving at the speed of light for one Planck length and one Planck time, resulting in an average velocity of $0.1c$. Therefore, continuous velocity and discrete velocity are effectively indistinguishable.

The postulate of discrete spacetime and its implications would imply that an observable length contraction would have to be at least one Planck length long. This means that for a length contraction to be observable, the difference between an object's contracted and uncontracted length would have to be at least a Planck length. The question then becomes "What is the minimum velocity that an object of length $L_0 = n_0 l_p$, where $n_0 \geq 1$, can have in order for it to contract by a minimum length of one Planck length?" The derivation for this is shown below, and we shall see that it has a major implication. Starting with the length contraction equation, we have

$$L = \frac{L_0}{\gamma} = L_0\sqrt{1-\beta^2} \tag{65}$$

where L is the contracted length and L_0 is the uncontracted length. The minimum difference between the two leads to

$$\Delta l = L_0 - L = L_0(1 - \sqrt{1-\beta^2}) = 1l_p \tag{66}$$

$$1l_p = n_0 l_p(1 - \sqrt{1-\beta^2})$$

$$1 = n_0(1 - \sqrt{1-\beta^2}) = n_0(1 - \frac{\sqrt{c^2 - v^2}}{c})$$

$$= \frac{n_0}{c}(c - \sqrt{c^2 - v^2})$$

$$c - \frac{c}{n_0} = \sqrt{c^2 - v^2} \tag{67}$$

Solve equation (65) for v and we get

$$\boxed{v = \frac{c\sqrt{(2n_0 - 1)}}{n_0}} \tag{68}$$

This answers the above question, becoming the minimum velocity that an object would have to have in order to contract by a minimum observable length of one Planck length. We can generalize this equation to apply for any amount of observable contraction (i.e., $\Delta l = nl_p, n \geq 1$):

$$nl_p = n_0 l_p(1 - \sqrt{1-\beta^2})$$

$$n = n_0(1 - \frac{\sqrt{c^2 - v^2}}{c})$$

$$n = \frac{n_0}{c}(c - \sqrt{c^2 - v^2})$$

$$c - \frac{nc}{n_0} = \sqrt{c^2 - v^2}$$

$$v^2 = (c - \frac{nc}{n_0})^2 - c^2$$

$$\Rightarrow \boxed{v = \frac{c\sqrt{(2n_0 n - n^2)}}{n_0}} \tag{69}$$

where $n, n_0 \geq 1$ and

$$n_0 \equiv \text{how large } L_0 \text{ is to } l_p$$
$$n \equiv \text{how much } L_0 \text{ contracts by in terms of } l_p$$

From equation (68), we can see that an object of size one Planck length would have to move at the speed of light in order for any length contraction to be observable, and if such object contracts, then it would contract to be observed as having no lengths at all, making the object not observable. We also know that length contraction does not apply for the speed of light, and therefore we can arguably conclude that length contraction does not apply for the Planck length. This preserves the Planck length as being the minimum observable length.

A closer look at equation (69) reveals that there appears to be a major implication. Equation (69) reveals that the observable velocity an object can have is dependent on its size and how much it contracts by. The larger its size, the greater its velocity can be and the more it can contract. Near the Planck scale, the values of n and n_0 can only take on discrete values in order for any length or length contraction to be observable. This discreteness may still be true at the macroscopic scale where n and n_0 would take on much larger values. The reason why this is still the case is simply because n and n_0 comes from the normalization of Δl and L_0, respectively, via dividing both by l_p. Considering how small the Planck length is at the macroscopic scale, verifying that Δl, L_0, and L are all discretely proportional to l_p would be extremely difficult and would require extremely accurate measurements. Even a small amount of systematic or random error could affect this discreteness. In fact, it could be argued that the smallest amount of error in measuring the Planck length would be equal to that of the Planck length, which effectively would also preserve the Planck length as being the minimum length when it comes to actual experimental measurement.

Thus far we have seen that by taking relative locality into account, we can change the Lorentz factor in such a way as to see how it might preserve the Planck length. If we applied this to the Lorentz transformation, then likewise the transformations may or may not differ by a factor similar to those of k_x, k'_x, k_t, k'_t. We have also seen that taking the postulates of LQG into account would lead us to a discrete spacetime, where the velocity of an object may only take on certain values depending on its size or length. These values are represented by n and n_0 in equation (67), and although it will not be explored here, it might be the same as or similar to the wavenumber of a particle. Both approaches have shown or at least argued for a way to preserve the Planck length and by extension the preservation of Lorentz invariance at the Planck scale.

Discussion

After string theory, Loop Quantum Gravity is the second most well known theory trying to unite general relativity and quantum mechanics. It came about after many roadblocks in formulating a successful theory of quantum gravity, such as the canonical approach failing to realize the importance of elementary particles in quantum gravity. So far loop quantum gravity has also run into many roadblocks. Two major ones are the violation of Lorentz invariance and the consistent inclusion of matter. Lorentz invariance is what is responsible for length contraction ($L = L_0/\gamma$) and time dilation ($\Delta t' = \gamma \Delta t$), with the Lorentz factor: $\gamma = 1/\sqrt{1 - v^2/c^2}$. The violation of Lorentz invariance basically comes about when one considers the Planck length or the Planck scale. Being the smallest possible length, the Planck length theoretically should not shrink any further. However, classical length contraction says otherwise. The mitigation of this problem has been attempted via constraining Lorentz violation to occur at a scale much higher than the Planck scale. This can be done via an Effective Field Theory or matter.

In the case of an Effective Field Theory, a modified dispersion relation points to a procedure of how to constrain when and where Lorentz violation may occur. This has pointed to the use of ultra-high energy cosmic rays (UHECRs) as a limit. We have for example the GZK limit where photons from UHECRs are expected to have a maximum energy of around $10^{11} eV/c^2$, which is no where near the Planck scale energy of about $10^{19} GeV/c^2$.

In the case of matter, no one knows of exactly how matter arises in quantum gravity. This is on top of the fact that matter certainly exists at the quantum scale and is usually responsible for the observation of gravity. So one attempt for this is by using scalar fields, since matter or mass is usually thought of as a scalar, and then coupling gravity to that scalar field. Considering that fundamental particles, such as the electron, are seen to have sizes (via their wavelengths) no where near the Planck scale, this would constrain Lorentz invariance to unlikely be violated. Regardless of how matter comes about, the inclusion of matter in quantum gravity will be necessary since it is through the utilization of matter in experiments and observations that measurements are made.

If these constraints are not enough to preserve Lorentz invariance at the Planck scale, then the theory of special relativity may need to be reformulated to not only preserve the speed of light but also the Planck length. This has led to the theory of deformed (or doubly) special relativity. Here we have seen that there are different models relying on a modification of their dispersion relation. This modification usually comes in the form of a function, such as $f(E, p; \kappa^{-1})$, that has the Planck length represented via a scale factor, κ^{-1}. Although deformed special relativity appears to preserve Lorentz invariance, it has made predictions, such as a varying speed of light, that has not held up to observations from gamma-ray bursts.

From the idea of a modified theory of special relativity, we have instead attempted to modify the Lorentz factor so as to see how it would preserve the Planck length. We have done this by utilizing the basic idea of relative locality and the postulates of Loop Quantum Gravity. In using the idea of relative locality, we saw that the Lorentz factor γ changed mostly by the factors of k_x, k'_x, k_t, k'_t, which are proportionality factors between distance and time in different frames of reference. We also saw that β is not the same in all reference frames. Depending on the values and signs of these factors, along with β and β', this new γ could preserve the Planck length via length dilation. Although not explored, this result does sound like a Doppler effect where sometimes we can get a contraction (blue shift) or a dilation (red shift), depending on the values and signs of these factors.

If we insist that Lorentz invariance only applies in flat-spacetime and thus the Lorentz factor cannot be changed, then using the postulates of Loop Quantum Gravity where there is a minimum observable length l_p and minimum observable time t_p, we have seen that length contraction and time dilation must come discretely. This has led to the requirement of an observable length contraction being at least one l_p, which resulted in the speed of a particle or object being dependent on its size. If an object is of one l_p then its minimum velocity is the speed of light in order for it to have any observable length contraction. We of course know that length contraction does not apply for the speed of light, and therefore can conclude that length contraction would not apply for the Planck length.

Future Work

The next steps in this work would be to find a way of determining what values the factors of k_x, k'_x, k_t, k'_t can actually take. Do their sign and the sign of β and β' lead to a Doppler-like effect? This would require a deeper dive into how geometry and curvature affects length and time, and how energy affects curvature. All of this would lead to a closer study of the Einstein field equations and how they may be changed to better fit a quantized spacetime. Likewise, we would also need to study what values n and n_0 in equation (69) can take and how are they related to the wavenumber. If at the quantum scale, these values can only be discrete then that could mean velocity may only take on discrete values as well.

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