# UNITAL LATTICE-ORDERED ALGEBRAS WITH A d-BASIS CONTAINING FOUR ELEMENTS

by

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DEDICATION

This thesis is dedicated to my parents

**Mr. Phu Tran**

**Mrs. Hoang Nguyen**

for all their unconditional love, confidence and sacrifices for their children

to my dear brother

**NhuVu (Bi) Tran**

for his wholehearted encouragement

and to my lovely sweetheart

**Mai (Vivian) Huynh**

for her passionate love, endless supports and enthusiastic cheers

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**who introduced me to mathematical research and energetically**
**taught me the sophistication of abstract algebra**

ABSTRACT

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An investigation is constructed and implemented for all unital lattice-ordered algebras with a *d*-basis containing 4 elements. All such lattice-ordered algebras will be expected to be assembled using 2x2 matrices, group algebras, lattice-ordered fields, direct sums and/or combinations of them.

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# 1. INTRODUCTION

## 1.1. The Problem.

The purpose of this thesis is to classify all 4-dimensional $\ell$-algebras with a $d$-basis. In other words, we want to know what a 4-dimensional $\ell$-algebra with a $d$-basis looks like. Some examples of such algebra are the $2 \times 2$ matrices, 4-dimensional group algebras, and 4-dimensional fields with the usual lattice orders. We are going to use $2 \times 2$ matrix $\ell$-algebra, semigroup $\ell$-algebras, and $\ell$-fields as building blocks to construct all 4-dimensional $\ell$-algebras with a $d$-basis.

Let $A$ be a unital 4-dimensional $\ell$-algebras over a totally ordered field $F$ with a $d$-basis. Let $D = \{d_1, d_2, d_3, d_4\}$ be a $d$-basis for $A$ and 1 be the identity element of $A$. Then, 1 is the positive linear combination of elements in $D$. So we have 4 cases to be considered.

a. $1 = \alpha_1 d_1 + \alpha_2 d_2 + \alpha_3 d_3 + \alpha_4 d_4$, $0 < \alpha_i \in F$.
b. $1 = \alpha_1 d_1 + \alpha_2 d_2 + \alpha_3 d_3$, $0 < \alpha_i \in F$.
c. $1 = \alpha_1 d_1 + \alpha_2 d_2$, $0 < \alpha_i \in F$.
d. $1 = \alpha_1 d_1$, $0 < \alpha_1 \in F$.

Each case includes many other subcases, and we, certainly, will cover all possible situations according to each case by using the building blocks we stated above.

## 1.2. Historical Context.

G. Birkhoff and R.S. Pierce are the pioneers in the research of lattice-ordered rings ($\ell$-rings) and lattice-ordered algebras ($\ell$-algebras). Their first paper [1] published in 1956 set the stage for the research in this area that continues to this day. Due to the difficulty in the study of general theory. they recommended to study some special classes of $\ell$-rings and $\ell$-algebras. One of the special classes - the class of function rings ($f$-rings) - has been extensively studied. Although the theory of $f$-rings is very useful and applicable, it is missing some important algebraic structures, for instances, matrices and polynomials with standard lattice order. Hence, more studies need to be done in order to include those examples and of course, to provide a more general structure theory for $\ell$-rings not being $f$-rings. In attempt of doing so, M. Henriksen, in Problem 4 of [3], presented the following problem:

> "Develop a structure theory for a class of lattice-ordered rings that include semigroup algebras over $\mathbb{R}$ ordered as follows. If $S$ is a multiplicative semigroup, $s_1, s_2, ..., s_k \in S$, and $a_1, a_2, ..., a_n \in \mathbb{R}$, let $\sum a_i s_i \geq 0$ if $a_i \geq 0$ for $1 \leq i \leq n$. Do this at least for a class of semiproups large enough to include 1, x, ..., $x_n$, ... and the semigroup of unit of matrices $\{E_{ij}\}$ (where $E_{ij}$ has a 1 in row i and column j, and zeros elsewhere for $1 \leq i \leq n$ and $1 \leq j \leq n$)."

Recently, J. Ma introduces the class of $\ell$-algebras with a $d$-basis to study [4]. This class of $\ell$-algebras includes all examples mentioned above. Some good structure theories are expected to obtain for this class of $\ell$-algebras. In [4], all the 2-dimensional and 3-dimensional $\ell$-algebras with a $d$-basis have been constructed. In his research project [2], A. Elkass studied 4-dimensional $\ell$-algebras with a $d$-basis; however, his research is restricted only to $\ell$-reduced cases. The $\ell$-reduced condition leaves one of the most important algebraic structures - the $2 \times 2$ matrices - out of consideration.

In this thesis, I will construct all possible 4-dimensional $\ell$-algebras without $\ell$-reduced condition. This will give us the insight of the general structure of finite-dimensional $\ell$-algebras with a $d$-basis and provide the supporting evidences for such $\ell$-algebras having a good algebraic structure. It is a valuable step forward for future study of $\ell$-algebras.

# 2. FUNDAMENTAL CONCEPTS AND DEFINITIONS

## 2.1. Fundamental Concepts.

Certainly, in our daily life, every problem that we face has its own identity and characteristic. Thus, solving them seems like an individual case study. It is an hectic work. Therefore, people always want to have a general method to solve all of them. It is the beauty of abstract algebra. Algebra gives us an ability to categorize similar behavior entities. Thus, if we totally comprehence a structure, then whatever entities that satisfy all the requirement of that structure will have all properties of that structure. That is why people want to study structure theory. It would give us a general view.

Why do we study order? The answer is simple. Order is involved with everything in our daily life. For instance, we always want to compare things that we have. Some entities are simple to compare. For instance, on the number line, no matter where we pick up two points, we know which one is bigger. However, in many other cases, the number line rule does not apply. Let us pick up 2 points on the $xy$-plane, $A = (1,4)$ and $B = (4,1)$. The $x$-axis rule says $B > A$, the $y$-axis rule says $A > B$, and the distance-from-origin rule says $A = B$. That is why we need to study order structure to handle such situations. Lattice order is one of many important orders and it is used to solve these problems.

## 2.2. Definitions.

Here, we provide a list of some basic definitions and termintologies that will be used later in the content. They could be found in [1] and [4].

**Definition 2.1.** Let $G$ be a nonempty set and $+$ is a binary operation on $G$. $(G,+)$ is called a *group* if

1. $a+b \in G, \forall a,b \in G$.
2. $(a+b)+c = a+(b+c), \forall a,b,c \in G$.
3. $\exists\, 0 \in G$, such that $a+0=a$ and $0+a=a$.
4. $\forall a \in G, \exists -a \in G$ such that $a+(-a)=0$ and $(-a)+a=0$.

A group $G$ is *abelian* or *commutative* if $a+b=b+a, \forall a,b \in G$.

**Definition 2.2.** Let $R$ be a nonempty set and $+$, $\cdot$ be binary operations on $R$. $(R,+,\cdot)$ is called a *ring* if

1. $(R,+)$ is an abelian group.
2. $(a\cdot b)\cdot c = a\cdot(b\cdot c), \forall a,b,c \in R$.
3. $(a+b)\cdot c = a\cdot c + b\cdot c$ and $c\cdot(a+b) = c\cdot a + c\cdot b, \forall a,b,c \in R$.

If $a\cdot b = b\cdot a, \forall a,b \in R$, then $R$ is a *commutative ring*. If $\exists\, 1 \in R$, such that $a\cdot 1 = a$ and $1\cdot a = a$, then $R$ is a ring with identity 1.

**Definition 2.3.** A *partially order* on a set $P$ is a relation $\leq$ (i.e. $\leq \subseteq P\times P$) such that,

1. $p \leq p, \forall p \in P$.
2. $p \leq q$ and $q \leq p \Rightarrow p = q$.
3. $p \leq q, q \leq r \Rightarrow p \leq r$.

If a partial order satisfies the additional condition, $\forall p,q \in P \Rightarrow p \leq q$ or $q \leq p$, then, it is called the *total order*.

**Definition 2.4.** Let $P$ be a nonempty set and $\leq$ be a partial order on $P$. Then $(P, \leq)$ is called a *poset*. A poset $(P, \leq)$ is called a *lattice* if any two elements $a, b$ of $P$ have a least upper bound $a \vee b$, and the greatest lower bound $a \wedge b$. Then the order, $\leq$, is called a *lattice order*.

**Definition 2.5.** Let $(G, +)$ be a group and $(G, \leq)$ be a poset. G is called a *po-group* if

$$x \leq y \Rightarrow a + x + b \leq a + y + b, \forall a, b, x, y \in G.$$

A po-group is a *lattice-ordered* group ($\ell$-group) if the order $\leq$ is a lattice order. Let $G$ be a po-group. The *positive cone* of $G$ is defined as $G^{\geq} = \{g \mid g \geq 0\}$.

**Definition 2.6.** Given a ring $R(+, \cdot, \geq)$. $R$ is called a *po-ring* if,

1. $R(+, \geq)$ is a po-group.
2. $\forall a, b \in R, a \geq 0, b \geq 0 \Rightarrow a \cdot b \geq 0$.

Suppose $R$ is a ring. If a subset $P \subseteq R$ satisfies the following three conditions.

1. $P + P \subseteq P$,
2. $P \cdot P \subseteq P$,
3. $P \cap -P = \{0\}$, where $-P = \{x \in R \mid -x \in P\}$,

then $R(+, \cdot, \geq)$ is a po-ring with respect to the partial order, $\geq$, defined by $a \geq b \Leftrightarrow a - b \in P$ and $P = R^{\geq}$.

A po-ring $R(+, \cdot, \geq)$ is a *lattice-ordered ring* ($\ell$-ring) if $\geq$ is a lattice order.

A *lattice-ordered algebra* ($\ell$-algebra) $A$ over a totally ordered field $F$ is an algebra over $F$ which is an $\ell$-ring and $\forall \alpha \in F^{\geq}, a \in A^{\geq} \Rightarrow \alpha a \in A^{\geq}$.

For the rest of this thesis, $F$ always denotes a totally ordered subfield of the field $\mathbf{R}$ of real numbers.

**Definition 2.7.** Let $A$ be an $\ell$-algebra over $F$.

For $a \in A$, $a$ is called a *basic* element of $A$ if $a > 0$ and $\forall b \geq 0$ and $c \geq 0$, $a \geq b$ and $a \geq c \Rightarrow b$ and $c$ are comparable (i.e. either $b \geq c$ or $c \geq b$.)

For $x, y \in A$, $x$ and $y$ are called *disjoint* if $x > 0, y > 0$ and $x \wedge y = 0$.

An element $a$ of $A$ is called an *$f$-element* if

$$a \geq 0 \text{ and } \forall b, c \in A, b \wedge c = 0 \Rightarrow ab \wedge c = ba \wedge c = 0,$$

and $a$ is called a *$d$-element* if

$$a \geq 0 \text{ and } \forall b, c \in A, b \wedge c = 0 \Rightarrow ab \wedge ac = ba \wedge ca = 0.$$

**Definition 2.8.** Let $A$ be an $\ell$-algebra over $F$. $S$ is a nonempty subset of $A$. $S$ is called a *$d$-basis* of $A$ if the following conditions are satisfied.

1. $S$ is disjoint. (i.e. $\forall s_i, s_j \in S, s_i \neq s_j, s_i$ and $s_j$ are disjoint).
2. $A = \text{Span}(S)$ as a vector space over $F$.
3. $s_i$ is a $d$-element, $\forall s_i \in S$.

**Definition 2.9.** Let $A$ be an $\ell$-algebra over $F$.

$A$ is *unital* means that $A$ has an identity element, and $A$ is *$\ell$-unital* means that $A$ has a positive identity element. It was proved that a unital $\ell$-algebra with a $d$-basis must be $\ell$-unital [1].

$A$ is *$\ell$-reduced* means that $A$ has no nonzero positive nilpotent elements.

**Definition 2.10.** (1) Let $M_n(F)$ $(n \geq 2)$ be the $n \times n$ matrix algebra over $F$. With the positive cone $M_n(F)^{\geq} = M_n(F^{\geq})$, $M_n(F)$ becomes an $\ell$-algebra.

(2) Similarly, let $T_n(F)$ $(n \geq 2)$ be the $n \times n$ upper triangular matrix algebra over $F$. With the positive cone $T_n(F)^{\geq} = T_n(F^{\geq})$, $T_n(F)$ becomes an $\ell$-algebra.

(3) Let $0 < \alpha \in \mathbf{R}$ and $\sqrt{\alpha} \notin F$. Then $F(\sqrt{\alpha}) = \{a + b\sqrt{\alpha} : a, b \in F\}$ becomes an $\ell$-field with the positive cone $F(\sqrt{\alpha})^{\geq} = \{a + b\sqrt{\alpha} : a, b \in F^{\geq}\}$.

(4) Let $G$ be a group and $F[G]$ be the group algebra of $G$ over $F$. Then $F[G]$ becomes an $\ell$-algebra over $F$ if we define an element $\sum \alpha_i g_i \geq 0$, where $\alpha_i \in F, g_i \in G$, if and only if each $\alpha_i \geq 0$.

In this thesis, by matrix $\ell$-algebras, upper triangular matrix $\ell$-algebras, $\ell$-fields, and group $\ell$-algebras, we mean the lattice order defined above.

## 3. PROPOSITIONS

Here, we list and prove some propositions that will be used later to constract lattice orders on a 4-dimensional algebra $A$ over $F$ with a $d$-basis.

**Proposition 3.1.** *Let $D = \{d_1, d_2, d_3, d_4\}$ be a $d$-basis of $A$ over $F$ and 1 be the identity of $A$. If $1 = \alpha_1 d_1 + \alpha_2 d_2 + \alpha_3 d_3 + \alpha_4 d_4$, then $D^* = \{a_1, a_2, a_3, a_4\}$ is also a $d$-basis of $A$, where $a_i = \alpha_i d_1$, $i = 1, 2, 3, 4$.*

*Proof.* See Case 1 in Theorem 1.7 of [2]. □

By Proposition 3.1, for a unital 4-dimensional $\ell$-algebra with a $d$-basis, we may assume that there exists a $d$-basis $D = \{d_1, d_2, d_3, d_4\}$ such that the identity element 1 is a sum of some elements in $D$.

**Proposition 3.2.** *Let $D = \{d_1, d_2, d_3, d_4\}$ be a $d$-basis of $A$ over $F$ and 1 be the identity of $A$ such that $1 = \sum_{i=1}^{n} d_i$, $2 \leq n \leq 4$.*

(1) *For each $1 \leq k \leq n$, $d_k^2 = d_k$, and for any $1 \leq r, s \leq n$, $r \neq s$, $d_r d_s = 0$.*
(2) *For each $d \in D$, $d = d_k d$ for some $1 \leq k \leq n$ and $dd_t = 0$ for any $1 \leq t \leq n$ and $t \neq k$. Similarly $d_u d = d$ for some $1 \leq u \leq n$ and for any $1 \leq v \leq n$, $v \neq u$, $d_v d = 0$.*

*Proof.* (1) From $1 = \sum_{i=1}^{n} d_i, n = 2, 3, 4$, $0 \leq d_k \leq 1$ for each $d_k$ $1 \leq k \leq n$, so each $d_k$ in the sum of 1 is an $f$-element. Then $d_r \wedge d_s = 0$ implies $d_r d_s \wedge d_r d_s = 0$, so $d_r d_s = 0$ for any $1 \leq r, s \leq n$, $r \neq s$. Thus $d_k^2 = d_k$ for each $1 \leq k \leq n$.

(2) Let $d \in D$. Then $d = \sum_{i=1}^{n} dd_i$. Since $d$ is basic, any two elements in this sum are comparable. On the other hand, since $d$ is a $d$-element, $dd_r \wedge dd_s = 0$ for any $r \neq s$. Therefore, there exists only one term in the sum of $d$ is not zero. This completes the proof. □

Let $A$ be a unital 4-dimensional $\ell$-algebra with a $d$-basis $D$. In the following, $A$ will be constructed by constructing a multiplication table for elements in $D$. One important property of $D$ is that $D$ is *cancellative* as stated in the following result.

**Proposition 3.3.** *Let $A$ be a unital $\ell$-algebra with a $d$-basis $D$. The $D$ is cancellative in the sense that for any $r, s, t \in D$,*

$$rs = rt \neq 0 \Rightarrow s = t \text{ and } sr = tr \neq 0 \Rightarrow s = t.$$

*Proof.* Suppose that $rs = rt \neq 0$. If $s \neq t$, then $s \wedge t = 0$. Since $r$ is a $d$-element, $rs \wedge rt = 0$, so $rs = rt = rs \wedge rt = 0$, which is a contradiction. □

As a direct corollary of Proposition 3.3, in a multiplication table for elements in a $d$-basis $D$, any row and any column do not contain nonzero repeated element. This situation is similar to the multiplication table of a finite group.

## 4. RESEARCH AND RESULTS

In this section we construct all 4-dimensional $\ell$-algebra over $F$ with a $d$-basis. There are some $\ell$-reduced cases that have been proven in [2]. Therefore, when those situations are encountered, I only list the results. This paper only proves the cases other than $\ell$-reduced ones. Let $A$ be a unital 4-dimensional $\ell$-algebras with a $d$-basis $D$. We assume that $D = \{a_1, a_2, a_3, a_4\}$ and 1 be the identity element in $A$. By Proposition 3.1, we may assume that 1 is the sum of some elements in $D$. We consider the following 4 cases according to how many elements in the $d$-basis $D$ is in the sum of 1.

### 4.1. Case 1: $1 = a_1 + a_2 + a_3 + a_4$. .

Although this case was covered in [2], I would like to restate it with a purpose of letting readers get used to the way all other cases are constructed. Let us exam the following.

From $1 = a_1 + a_2 + a_3 + a_4$ and Proposition 3.2, we have

$$a_i^2 = a_i \text{ and } a_i a_j = 0, \text{ for } 1 \leq i, j \leq 4, i \neq j.$$

Then we have the following multiplication table.

| | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
|---|---|---|---|---|
| $a_1$ | $a_1$ | 0 | 0 | 0 |
| $a_2$ | 0 | $a_2$ | 0 | 0 |
| $a_3$ | 0 | 0 | $a_3$ | 0 |
| $a_4$ | 0 | 0 | 0 | $a_4$ |

So, $A \cong F \oplus F \oplus F \oplus F$ is a 4-dimensional $f$-algebra.

### 4.2. Case 2: $1 = a_1 + a_2 + a_3$.

Suppose that $D = \{a_1, a_2, a_3, d_4\}$ is a $d$-basis in this case. By Proposition 3.2, there is only one $a_i d_4 \neq 0$, where $i = 1, 2, 3$. Without the loss of generality, we may assume that $a_1 d_4 \neq 0$. Hence, $a_2 d_4 = a_3 d_4 = 0$.

Similarly, we obtain $d_4 = d_4 a_1 + d_4 a_2 + d_4 a_3$. With the same argument, only one of $d_4 a_i$, $(i = 1, 2, 3)$, is not equal to zero.

4.2.1. $\underline{d_4 a_1 \neq 0}$. This implies that $d_4 a_1 = d_4$. So, we encounter 2 cases, which are $d_4^2 \neq 0$ and $d_4^2 = 0$. The first case is $\ell$-reduced case and was covered in [2], and the results are.

$$A \cong \begin{cases} F \oplus F \oplus F[G], \text{ where } G \text{ is a cyclic group of order 2;} \\ F \oplus F \oplus F(\sqrt{\lambda}), \text{ where } 0 < \lambda \in F, \sqrt{\lambda} \notin F, \text{ so } F(\sqrt{\lambda}) \text{ is an } \ell\text{-field.} \end{cases}$$

The second case provides us the following table.

| | $a_1$ | $d_4$ | $a_2$ | $a_3$ |
|---|---|---|---|---|
| $a_1$ | $a_1$ | $d_4$ | 0 | 0 |
| $d_4$ | $d_4$ | 0 | 0 | 0 |
| $a_2$ | 0 | 0 | $a_2$ | 0 |
| $a_3$ | 0 | 0 | 0 | $a_3$ |

Thus

$$A \cong (F1 \oplus Fa) \oplus F \oplus F,$$

where $a^2 = 0, 1a = a1 = a$.

4.2.2. $d_4a_2 \neq 0$. This is not a $\ell$-reduced case, because $d_4^2 = 0$. Here is the reason. Since $d_4a_2 \neq 0$, $d_4 = d_4a_2$. Then $d_4^2 = d_4a_2d_4a_2 = 0$ because $a_2d_4 = 0$. Here is the multiplication table.

| | $a_1$ | $a_2$ | $d_4$ | $a_3$ |
|---|---|---|---|---|
| $a_1$ | $a_1$ | 0 | $d_4$ | 0 |
| $a_2$ | 0 | $a_2$ | 0 | 0 |
| $d_4$ | 0 | $d_4$ | 0 | 0 |
| $a_3$ | 0 | 0 | 0 | $a_3$ |

We notice that

| | $a_1$ | $a_2$ | $d_4$ |
|---|---|---|---|
| $a_1$ | $a_1$ | 0 | $d_4$ |
| $a_2$ | 0 | $a_2$ | 0 |
| $d_4$ | 0 | $d_4$ | 0 |

$\cong$

| | $e_{11}$ | $e_{22}$ | $e_{12}$ |
|---|---|---|---|
| $e_{11}$ | $e_{11}$ | 0 | $e_{12}$ |
| $e_{22}$ | 0 | $e_{22}$ | 0 |
| $e_{12}$ | 0 | $e_{12}$ | 0 |

,

where $e_{ij}$ is a $2 \times 2$ matrix whose entries are 1 at $i^{th}$ row, $j^{th}$ column and 0 elsewhere. So

$$A \cong F \oplus T_2(F),$$

where $T_2(F)$ is the $2 \times 2$ upper triangular matrix $\ell$-algebra.

4.2.3. $d_4a_3 \neq 0$. We notice that under the assumotion in 4.2, $a_2$ and $a_3$ are summetric. Thus if we interchange $a_2$ and $o_3$, then this case is the same as 4.2.2.

4.3. **Case 3:** $1 = a_1 + a_2$.

We suppose that $D = \{a_1, a_2, d_3, d_4\}$ is a $d$-basis. Then we have the multiplication table.

| | $a_1$ | $a_2$ | $d_3$ | $d_4$ |
|---|---|---|---|---|
| $a_1$ | $a_1$ | 0 | $a_1d_3$ | $a_1d_4$ |
| $a_2$ | 0 | $a_2$ | $a_2d_3$ | $a_2d_4$ |
| $d_3$ | $d_3a_1$ | $d_3a_2$ | $d_3^2$ | $d_3d_4$ |
| $d_4$ | $d_4a_1$ | $d_4a_2$ | $d_4d_3$ | $d_4^2$ |

We recognize that there are four possible cases. They are

(1) $d_3^2 \neq 0$, $d_4^2 \neq 0$,
(2) $d_3^2 = 0$, $d_4^2 \neq 0$,
(3) $d_3^2 \neq 0$, $d_4^2 = 0$,
(4) $d_3^2 = 0$, $d_4^2 = 0$.

The first possibility constitutes an $\ell$-reduced case, and the others do not. In the non $\ell$-reduced situations, the first 2 cases are interchangeable by rearranging $d_3$ and $d_4$. Therefore, we only consider the cases in which $d_3^2 \neq 0$, $d_4^2 = 0$ and $d_3^2 = 0$, $d_4^2 = 0$.

4.3.1. $d_3^2 \neq 0$, $d_4^2 \neq 0$. This is an $\ell$-reduced case covered in [2]. $A$ is isomorphic to one of the following.

i. $F[G] \oplus F[G]$, where $G$ is a cyclic groups of order 2.

ii. $F[G] \oplus F(\sqrt{\lambda})$, where $G$ is a cyclic group of order 2, $0 < \lambda \in F$ but $\sqrt{\lambda} \notin F$, and $F(\sqrt{\lambda})$ is an $\ell$-field.

iii. $F(\sqrt{\lambda}) \oplus F(\sqrt{\lambda})$, where $0 < \lambda \in F$ but $\sqrt{\lambda} \notin F$, and $F(\sqrt{\lambda})$ is an $\ell$-fields.

iv. $F \oplus F[G]$, where $G$ is a cyclic group of order 3.

v. $F \oplus F(\sqrt[3]{\lambda})$, where $0 < \lambda \in F$ but $\sqrt[3]{\lambda} \notin F$ and $F(\sqrt[3]{\lambda})$ is an $\ell$-field.

4.3.2. $d_3^2 \neq 0$, $d_4^2 = 0$. In this case, we need to consider all the following 4 cases.

(a). $a_1d_3 = 0, a_2d_3 = d_3; a_1d_4 = 0, a_2d_4 = d_4$.

We obtain the table below.

| | $a_1$ | $a_2$ | $d_3$ | $d_4$ |
|---|---|---|---|---|
| $a_1$ | $a_1$ | 0 | 0 | 0 |
| $a_2$ | 0 | $a_2$ | $d_3$ | $d_4$ |
| $d_3$ | $d_3a_1$ | $d_3a_2$ | $d_3^2$ | $d_3d_4$ |
| $d_4$ | $d_4a_1$ | $d_4a_2$ | $d_4d_3$ | 0 |

Let us determine the rest values in the table.

$$
\begin{aligned}
d_3^2 = d_3(a_2d_3) = (d_3a_2)d_3 \neq 0 &\Rightarrow d_3a_2 = d_3 \text{ and } d_3a_1 = 0.\\
d_3^2 = \alpha_1a_1 + \beta_1a_2 + \lambda_1d_4 &\Rightarrow a_2d_3^2 = \alpha_1a_2a_1 + \beta_1a_2^2 + \lambda_1a_2d_4\\
&\Rightarrow d_3^2 = \beta_1a_2 + \lambda_1d_4.\\
d_3d_4 = \alpha_2a_1 + \beta_2a_2 &\Rightarrow d_3^2d_4 = \alpha_2d_3a_1 + \beta_2d_3a_2\\
&\Rightarrow d_3^2d_4 = \beta_2d_3.\\
&\Rightarrow (\beta_1a_2 + \lambda_1d_4)d_4 = \beta_2d_3\\
&\Rightarrow \beta_1a_2d_4 + \lambda_1d_4^2 = \beta_2d_3\\
&\Rightarrow \beta_1d_4 = \beta_2d_3.\\
&\Rightarrow \beta_1 = \beta_2 = 0\\
&\Rightarrow d_3^2 = \lambda_1d_4, \lambda_1 > 0\\
d_3d_4 = \alpha_2a_1 &\Rightarrow a_2d_3d_4 = \alpha_1a_2a_1.\\
&\Rightarrow d_3d_4 = 0.\\
d_3^2a_1 = \lambda_1d_4a_1 &\Rightarrow 0 = \lambda_1d_4a_1\\
&\Rightarrow d_4a_1 = 0, \text{ and } d_4a_2 = d_4.\\
d_4d_3 = \alpha_3a_1 + \beta_3a_2 &\Rightarrow d_3d_4d_3 = \alpha_3d_3a_1 + \beta_3d_3a_2\\
&\Rightarrow 0 = \beta_3d_3, \text{ so } \beta_3 = 0\\
&\Rightarrow d_4d_3 = \alpha_3a_1\\
&\Rightarrow a_1d_4d_3 = \alpha_3a_1^2\\
&\Rightarrow 0 = \alpha_3a_1\\
&\Rightarrow \alpha_3 = 0\\
&\Rightarrow d_4d_3 = 0.
\end{aligned}
$$

Let $d = \lambda_1d_4$, we have the following multiplication table.

| | $a_1$ | $a_2$ | $d_3$ | $d$ |
|---|---|---|---|---|
| $a_1$ | $a_1$ | 0 | 0 | 0 |
| $a_2$ | 0 | $a_2$ | $d_3$ | $d$ |
| $d_3$ | 0 | $d_3$ | $d$ | 0 |
| $d$ | 0 | $d$ | 0 | 0 |

Therefore,

$$A \cong F \oplus (F1 \oplus Fa \oplus Fa^2), \text{ where } a^3 = 0.$$

**(b).** $\underline{a_1d_3 = 0, a_2d_3 = d_3; a_1d_4 = d_4, a_2d_4 = 0.}$

We obtain the table below.

| | $a_1$ | $a_2$ | $d_3$ | $d_4$ |
|---|---|---|---|---|
| $a_1$ | $a_1$ | 0 | 0 | $d_4$ |
| $a_2$ | 0 | $a_2$ | $d_3$ | 0 |
| $d_3$ | $d_3a_1$ | $d_3a_2$ | $d_3^2$ | $d_3d_4$ |
| $d_4$ | $d_4a_1$ | $d_4a_2$ | $d_4d_3$ | 0 |

Again, let us determine the rest values in the table.

$$\begin{aligned}
d_3^2 = d_3(a_2d_3) &\Rightarrow \underline{d_3a_2 = d_3} \text{ and } \underline{d_3a_1 = 0}.\\
d_3^2 = \alpha_1a_1 + \beta_1a_2 + \lambda_1d_4 &\Rightarrow a_2d_3^2 = \alpha_1a_2a_1 + \beta_1a_2^2 + \lambda_1a_2d_4\\
&\Rightarrow \underline{d_3^2 = \beta_1a_2}, \beta_1 > 0.\\
d_3d_4 = (d_3a_2)(a_1d_4) &\Rightarrow \underline{d_3d_4 = 0}.\\
d_4d_3 = \alpha_2a_1 &\Rightarrow d_4d_3d_4 = \alpha_2a_1d_4\\
&\Rightarrow 0 = \alpha_2d_4, \text{ so } \alpha_2 = 0\\
&\Rightarrow \underline{d_4d_3 = 0}.\\
\beta_1d_4a_2 = d_4d_3^2 &\Rightarrow \beta_1d_4a_2 = 0\\
&\Rightarrow \underline{d_4a_2 = 0}, \text{ and } \underline{d_4a_1 = d_4}.
\end{aligned}$$

Here is the new table with $\alpha = \beta_1 > 0$.

| | $a_1$ | $d_4$ | $a_2$ | $d_3$ |
|---|---|---|---|---|
| $a_1$ | $a_1$ | $d_4$ | 0 | 0 |
| $d_4$ | $d_4$ | 0 | 0 | 0 |
| $a_2$ | 0 | 0 | $a_2$ | $d_3$ |
| $d_3$ | 0 | 0 | $d_3$ | $\alpha a_2$ |

Therefore,

$$A \cong \begin{cases} (F1 \oplus Fa) \oplus F[G], & \text{where } a^2 = 0,\ G \text{ is a cyclic group of order 2 } (\sqrt{\alpha} \in F);\\ (F1 \oplus Fa) \oplus F(\sqrt{\alpha}), & \text{where } a^2 = 0,\ F(\sqrt{\alpha}) \text{ is an } \ell\text{-field } (\sqrt{\alpha} \notin F). \end{cases}$$

**(c).** $\underline{a_1d_3 = d_3, a_2d_3 = 0; a_1d_4 = 0, a_2d_4 = d_4.}$

Since in 4.3 $a_1$ and $a_2$ are in a symmetric position, when we interchange $a_1$ and $a_2$, this case is the same as the case **(b)**.

**(d).** $\underline{a_1d_3 = d_3, a_2d_3 = 0; a_1d_4 = d_4, a_2d_4 = 0.}$

Similarly, if we interchange $a_1$ and $a_2$, this case becomes the case **(a)**.

4.3.3. $d_3^2 = 0$. $d_4^2 = 0$. First, let us determine the value of $d_3d_4$ and $d_4d_3$. Here is the table.

| | $a_1$ | $a_2$ | $d_3$ | $d_4$ |
|---|---|---|---|---|
| $a_1$ | $a_1$ | 0 | $a_1d_3$ | $a_1d_4$ |
| $a_2$ | 0 | $a_2$ | $a_2d_3$ | $a_2d_4$ |
| $d_3$ | $d_3a_1$ | $d_3a_2$ | 0 | $d_3d_4$ |
| $d_4$ | $d_4a_1$ | $d_4a_2$ | $d_4d_3$ | 0 |

Since $d_3$ and $d_4$ are interchangeable, we have 3 cases for product of $d_3$ and $d_4$: $d_3d_4 \neq 0$, $d_4d_3 \neq 0$; $d_3d_4 \neq 0$, $d_4d_3 = 0$; and $d_3d_4 = 0$, $d_4d_3 = 0$.

We claim that the second case is impossible. Suppose $d_3d_4 \neq 0$ and $d_4d_3 = 0$, then $d_3d_4 = \alpha_1a_1 + \beta_1a_2$. If we multiply $d_4$ to the previous equation, we have

$$d_4d_3d_4 = \alpha_1d_4a_1 + \beta_1d_4a_2 \Rightarrow 0 = \alpha_1d_4a_1 + \beta_1d_4a_2.$$

By Proposition 3.2, we have either $d_4a_1 = d_4$, $d_4a_2 = 0$, or $d_4a_1 = 0$, $d_4a_2 = d_4$. In the first case, we have $\alpha_1 = 0$ and $d_3d_4 = \beta_1a_2$. Then $0 = (d_3d_4)a_2 = \beta_1a_2$, so $\beta_1 = 0$, and hence $d_3d_4 = 0$. This is a contradiction. In the second case, we have $\beta_1 = 0$, so $d_3d_4 = \alpha_1a_1$. Then similarly, $0 = d_3d_4a_1 = \alpha_1a_1$, so $\alpha_1 = 0$, and hence $d_3d_4 = 0$ again, which is a contradiction.

Hence, we only have 2 cases, which are both $d_3d_4$ and $d_4d_3$ are equal to 0 or both are not equal to zero. In the following we consider those two cases.

(a). $d_3d_4 = d_4d_3 = 0$.

Here is the multiplication table.

| | $a_1$ | $a_2$ | $d_3$ | $d_4$ |
|---|---|---|---|---|
| $a_1$ | $a_1$ | 0 | $a_1d_3$ | $a_1d_4$ |
| $a_2$ | 0 | $a_2$ | $a_2d_3$ | $a_2d_4$ |
| $d_3$ | $d_3a_1$ | $d_3a_2$ | 0 | 0 |
| $d_4$ | $d_4a_1$ | $d_4a_2$ | 0 | 0 |

We notice that, the structure of the table is depend on the combination of the remaining upper right $2 \times 2$, which is called [R] and lower left $2 \times 2$, which is called [L]. By Proposition 3.2, we have,

$$[\mathrm{R}]: \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = A, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = B, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = C, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = D.$$

$$[\mathrm{L}]: \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = 1, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 3, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = 4.$$

So, we technically have 16 cases that are A1, ..., A4, B1, ..., B4, C1, ..., C4, D1, ..., D4. However, $a_1$ and $a_2$ are in symmetric position, so are $d_3$ and $d_4$. If we interchange their positions respectively, we obtain the same structures. By doing so, we have A1 = D4, A2 = A3 = D2 = D3, A4 = D1, B1 = B4. Therefore, we finally have only 6 cases. Here are all the cases.

A1. Here is the table

| | $a_2$ | $a_1$ | $d_3$ | $d_4$ |
|---|---|---|---|---|
| $a_2$ | $a_2$ | 0 | 0 | 0 |
| $a_1$ | 0 | $a_1$ | $d_3$ | $d_4$ |
| $d_3$ | 0 | $d_3$ | 0 | 0 |
| $d_4$ | 0 | $d_4$ | 0 | 0 |

Therefore,

$$A \cong F \oplus (F1 \oplus Fa \oplus Fb), \text{ where } a^2 = b^2 = ab = ba = 0.$$

**B2**. Here is the table

| | $a_1$ | $d_3$ | $a_2$ | $d_4$ |
|---|---|---|---|---|
| $a_1$ | $a_1$ | $d_3$ | 0 | 0 |
| $d_3$ | $d_3$ | 0 | 0 | 0 |
| $a_2$ | 0 | 0 | $a_2$ | $d_4$ |
| $d_4$ | 0 | 0 | $d_4$ | 0 |

Therefore,

$$A \cong (F1 \oplus Fa) \oplus (F1 \oplus Fb), \text{ where } a^2 = b^2 = 0.$$

**A2**. Here is the table

| | $a_2$ | $a_1$ | $d_4$ | $d_3$ |
|---|---|---|---|---|
| $a_2$ | $a_2$ | 0 | $d_4$ | $d_3$ |
| $a_1$ | 0 | $a_1$ | 0 | 0 |
| $d_4$ | 0 | $d_4$ | 0 | 0 |
| $d_3$ | $d_3$ | 0 | 0 | 0 |

Therefore,

$$A \cong T_2(F) \oplus Fa,$$

where $T_2(F)$ is the $2 \times 2$ upper triangular matrix $\ell$-algebra and $e_{11}a = ae_{11} = a, a^2 = 0$.

**A4**. Here is the table

| | $a_1$ | $a_2$ | $d_3$ | $d_4$ |
|---|---|---|---|---|
| $a_1$ | $a_1$ | 0 | $d_3$ | $d_4$ |
| $a_2$ | 0 | $a_2$ | 0 | 0 |
| $d_3$ | 0 | $d_3$ | 0 | 0 |
| $d_4$ | 0 | $d_4$ | 0 | 0 |

Therefore,

$$A \cong T_2(F) \oplus Fa$$

where $T_2(F)$ is the $2 \times 2$ upper triangular matrix $\ell$-algebra and $e_{11}a = ae_{22} = a, a^2 = 0$.

**B1**. Here is the table

| | $a_2$ | $a_1$ | $d_4$ | $d_3$ |
|---|---|---|---|---|
| $a_2$ | $a_2$ | 0 | $d_4$ | 0 |
| $a_1$ | 0 | $a_1$ | 0 | $d_3$ |
| $d_4$ | 0 | $d_4$ | 0 | 0 |
| $d_3$ | 0 | $d_3$ | 0 | 0 |

Therefore,

$$A \cong T_2(F) \oplus Fa,$$

where $T_2(F)$ is the $2 \times 2$ upper triangular matrix $\ell$-algebra and $e_{22}a = ae_{22} = a, a^2 = 0$.

B3. Here is the table

| | $a_1$ | $a_2$ | $d_3$ | $d_4$ |
|---|---|---|---|---|
| $a_1$ | $a_1$ | $0$ | $d_3$ | $0$ |
| $a_2$ | $0$ | $a_2$ | $0$ | $d_4$ |
| $d_3$ | $0$ | $d_3$ | $0$ | $0$ |
| $d_4$ | $d_4$ | $0$ | $0$ | $0$ |

Therefore.

$$A \cong T_2(F) \oplus Fa,$$

where $T_2(F)$ is the $2 \times 2$ upper triangular matrix $\ell$-algebra and $e_{22}a = ae_{11} = a, a^2 = 0$.

(b). $d_3d_4 \neq 0,\ d_4d_3 \neq 0$.

Here is the table.

| | $a_1$ | $a_2$ | $d_3$ | $d_4$ |
|---|---|---|---|---|
| $a_1$ | $a_1$ | $0$ | $a_1d_3$ | $a_1d_4$ |
| $a_2$ | $0$ | $a_2$ | $a_2d_3$ | $a_2d_4$ |
| $d_3$ | $d_3a_1$ | $d_3a_2$ | $0$ | $d_3d_4$ |
| $d_4$ | $d_4a_1$ | $d_4a_2$ | $d_4d_3$ | $0$ |

We may assume that $a_1d_3 = d_3$ and $a_2d_3 = 0$. Let $d_3d_4 = \alpha_1a_1 + \beta_1a_2$. Then

$$\begin{aligned} a_1d_3d_4 = \alpha_1a_1^2 + \beta_1a_1a_2 \ &\Rightarrow\ d_3d_4 = \alpha_1a_1, \alpha_1 > 0 \\ &\Rightarrow\ d_3d_4^2 = \alpha_1a_1d_4 \\ &\Rightarrow\ a_1d_4 = 0, a_2d_4 = d_4. \end{aligned}$$

Also

$$\begin{aligned} d_3d_4 = \alpha_1a_1, \alpha_1 > 0 \ &\Rightarrow\ d_3^2d_4 = \alpha_1d_3a_1 \\ &\Rightarrow\ d_3a_1 = 0, d_3a_2 = d_3. \end{aligned}$$

Suppose $d_4a_2 = d_4$, $d_4a_1 = 0$. Then $d_3d_4a_2 = d_3d_4 \Rightarrow \alpha_1a_1a_2 = d_3d_4 \Rightarrow 0 = d_3d_4$, which is a contradiction. So, $d_4a_1 = d_4$ and $d_4a_2 = 0$.

Let $d_4d_3 = \alpha_2a_1 + \beta_2a_2$ Then $d_4d_3a_2 = \alpha_2a_1a_2 + \beta_2a_2^2 \Rightarrow d_4d_3 = \beta_2a_2$. So, we obtain the following table.

| | $a_2$ | $a_1$ | $d_3$ | $d_4$ |
|---|---|---|---|---|
| $a_2$ | $a_2$ | $0$ | $0$ | $d_4$ |
| $a_1$ | $0$ | $a_1$ | $d_3$ | $0$ |
| $d_3$ | $d_3$ | $0$ | $0$ | $\alpha_1a_1$ |
| $d_4$ | $0$ | $d_4$ | $\beta_2a_2$ | $0$ |

Let us determine the relation between $\alpha_1$ and $\beta_2$. We have

$$\begin{aligned} d_4 d_3 = \beta_2 a_2 \quad &\Rightarrow \quad d_4 d_3 d_4 = \beta_2 a_2 d_4 \\ &\Rightarrow \quad d_4 \alpha_1 a_1 = \beta_2 d_4 \\ &\Rightarrow \quad \alpha_1 d_4 = \beta_2 d_4 \\ &\Rightarrow \quad \alpha_1 = \beta_2. \end{aligned}$$

Here is the table with $\alpha = \alpha_1 = \beta_2 > 0$.

| | $a_1$ | $a_2$ | $d_3$ | $d_4$ |
|---|---|---|---|---|
| $a_1$ | $a_1$ | $0$ | $0$ | $d_4$ |
| $a_2$ | $0$ | $a_2$ | $d_3$ | $0$ |
| $d_3$ | $d_3$ | $0$ | $0$ | $\alpha a_2$ |
| $d_4$ | $0$ | $d_4$ | $\alpha a_1$ | $0$ |

Now let $d = d_4/\alpha$. Then $\{a_1, a_2, d_3, d\}$ is a $d$-basis with the following multiplication table.

| | $a_1$ | $a_2$ | $d_3$ | $d$ |
|---|---|---|---|---|
| $a_1$ | $a_1$ | $0$ | $0$ | $d$ |
| $a_2$ | $0$ | $a_2$ | $d_3$ | $0$ |
| $d_3$ | $d_3$ | $0$ | $0$ | $a_2$ |
| $d$ | $0$ | $d$ | $a_1$ | $0$ |

Comparing the above multiplication table with the following multiplication table of the standard $2 \times 2$ matrix units,

| | $e_{11}$ | $e_{22}$ | $e_{21}$ | $e_{12}$ |
|---|---|---|---|---|
| $e_{11}$ | $a_1$ | $0$ | $0$ | $e_{12}$ |
| $e_{22}$ | $0$ | $e_{22}$ | $e_{21}$ | $0$ |
| $e_{21}$ | $e_{21}$ | $0$ | $0$ | $e_{22}$ |
| $e_{12}$ | $0$ | $e_{12}$ | $e_{11}$ | $0,$ |

we have

$$A \cong M_2(F),$$

where $M_2(F)$ is the $2 \times 2$ matrices $\ell$-algebra with the entrywise lattice order.

### 4.4. Case 4: $1 = a_1$.

This case was covered in [4] and [2]. Here are the results. $A$ is isomorphic to one of the following.

1. $F[G]$, where $G$ is a cyclic group of order 4.
2. $F[G]$, where $G = \{e, a, b, ab\}$ is a group defined by $a^2 = b^2 = e$ and $ab = ba$.
3. $F(b)$, where $0 < b \in \mathbb{R}\backslash F, b^4 \in F$, and $f(x) = x^4 - b^4$ is irreducible over $F$.
4. $F[G] \oplus F[G]b$, where $G$ is a cyclic group of order 2 and $b^2 = 1$.
5. $F(b) \oplus F(b)a$, where $0 < b \in \mathbb{R}\backslash F, b^2 \in F$ *and* $a^2 = b$.

# 5. CONCLUSION

**5.1. Summary.** We conclude that a unital 4-dimensional $\ell$-algebra $A$ with a $d$-basis is isomorphic to one of the following $\ell$-algebras.

1. $F \oplus F \oplus F \oplus F$.
2. $F \oplus F \oplus F[G]$, where G is a cyclic group of order 2.
3. $F \oplus F \oplus F(\sqrt{\lambda})$, where $0 < \lambda \in F$, $\sqrt{\lambda} \notin F$, so $F(\sqrt{\lambda})$ is an $\ell$-field.
4. $(F1 \oplus Fa) \oplus F \oplus F$, where $a^2 = 0, 1a = a1 = a$.
5. $F \oplus T_2(F)$, where $T_2(F)$ is the $2 \times 2$ upper triangular matrix $\ell$-algebra.
6. $F[G] \oplus F[G]$, where $G$ is a cyclic group of order 2.
7. $F[G] \oplus F(\sqrt{\lambda})$, where $G$ is a cyclic group of order 2, $0 \leq \lambda \in F$, $\sqrt{\lambda} \notin F$. and $F(\sqrt{\lambda})$ is an $\ell$-field.
8. $F(\sqrt{\lambda}) \oplus F(\sqrt{\lambda})$, where $0 \leq \lambda \in F$, $\sqrt{\lambda} \notin F$, and $F(\sqrt{\lambda})$ is an $\ell$-fields.
9. $F \oplus F[G]$, where $G$ is a cyclic group of order 3.
10. $F \oplus F(\sqrt[3]{\lambda})$, where $0 \leq \lambda \in F$, $\sqrt[3]{\lambda} \notin F$, and $F(\sqrt[3]{\lambda})$ is an $\ell$-field.
11. $F \oplus (F1 \oplus Fa \oplus Fa^2)$, where $a^3 = 0$.
12. $(F1 \oplus Fa) \oplus (F1 \oplus Fb)$, where $a^2 = b^2 = 0$.
13. $(F1 \oplus Fa) \oplus F[G]$, where $a^2 = 0$ and G is a cyclic group of order 2.
14. $(F1 \oplus Fa) \oplus F(\sqrt{\alpha})$, where $0 < \alpha \in F$ but $\sqrt{\alpha} \notin F$, so $F(\sqrt{\alpha})$ is an $\ell$-field and $a^2 = 0$.
15. $F \oplus (F1 \oplus Fa \oplus Fb)$, where $a^2 = b^2 = ab = ba = 0$.
16. $A \cong T_2(F) \oplus Fa$, where $T_2(F)$ is a $2 \times 2$ upper triangular matrix $\ell$-algebra and $e_{11}a = ae_{11} = a$.
17. $A \cong T_2(F) \oplus Fa$, where $T_2(F)$ is a $2 \times 2$ upper triangular matrix $\ell$-algebra and $e_{11}a = ae_{22} = a$.
18. $A \cong T_2(F) \oplus Fa$, where $T_2(F)$ is a $2 \times 2$ upper triangular matrix $\ell$-algebra and $e_{22}a = ae_{22} = a$.
19. $A \cong T_2(F) \oplus Fa$, where $T_2(F)$ is a $2 \times 2$ upper triangular matrix $\ell$-algebra and $e_{22}a = ae_{11} = a$.
20. $A \cong M_2(F)$, where $M_2(F)$ is the $2 \times 2$ matrices $\ell$-algebra.
21. $F[G]$, where $G$ is a cyclic group of order 4.
22. $F[G]$, where $G = \{e, a, b, ab\}$ is a group defined by $a^2 = b^2 = e$ and $ab = ba$.
23. $F(b)$, where $0 < b \in \mathbb{R} \backslash F, b^4 \in F$ and $f(x) = x^4 - b^4$ is irreducible over $F$.
24. $F[G] \oplus F[G]b$, where $G$ is a cyclic group of order 2 and $b^2 = 1$.
25. $F(b) \oplus F(b)a$, where $0 < b \in \mathbb{R} \backslash F, b^2 \in F$ and $a^2 = b$.

**5.2. Conclusion.** Half of a century has passed since the first systematical study of $\ell$-algebras; we still do not have a general structure theory for $\ell$-algebras. From the dawn of the research in this field, which was dated back to 1956, G. Birkhoff and R.S. Pierce already realized the difficulties in obtaining a general structure theory for $\ell$-algebras. Therefore, many scholars have tried to simply the problems by adding extra conditions, such as $f$-rings, $d$-rings, $\ell$-reduced, etc. Certainly, many great discoveries and papers had been found and published, but the general theory is still remaining mysterious. Undoubtedly, this research will not accomplish the ultimate goal, but it provides some supporting evidences for this arguement. As expected, this thesis includes the $2 \times 2$ matrices, which is left out in other papers. Nevertheless, it will provide to pave the road to success.

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