FUNDAMENTALS OF AXIS-SYMMETRIC BOUNDARY RECONSTRUCTION FOR IDEAL TOKAMAK PLASMAS: USING TOROIDAL HARMONICS TO MATCH POLOIDAL FLUX MEASUREMENTS IN THE SURROUNDING VACUUM

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ABSTRACT

FUNDAMENTALS OF AXIS-SYMMETRIC BOUNDARY RECONSTRUCTION FOR IDEAL TOKAMAK PLASMAS: USING TOROIDAL HARMONICS TO MATCH POLOIDAL FLUX MEASUREMENTS IN THE SURROUNDING VACUUM

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The University of Houston-Clear Lake, 2018

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With the development of tokamaks in the 1950s to control and sustain fusion reactions, a large amount of theory regarding associated plasmas has accumulated. Much of this theory is out of reach for the uninitiated, and applicable plasma physics texts either approach the field in too broad a path for specific reference, or they leave out many of the derivations needed for the foundational mechanics.

Here the fundamental math behind ideal tokamak plasmas is laid out to reconstruct the equilibrium boundary. This process illustrates the assumptions needed to idealize the mechanics, and thus gives wide avenues for alteration. Separation of variables is used in toroidal coordinates, and care is given to cover all the details. Optimizing the results to fit measurements from the surrounding vacuum can then produce a robust fixed boundary condition to be used in wider contexts.

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CHAPTER 1: OVERVIEW

Tokamaks are large machines used to contain and heat an ionized gas called a plasma to fusion temperatures. They were developed in the Soviet Union during the 1950s, and there has been a wealth of literature describing advances in their technology and usage ([1],[2],[3],etc...).

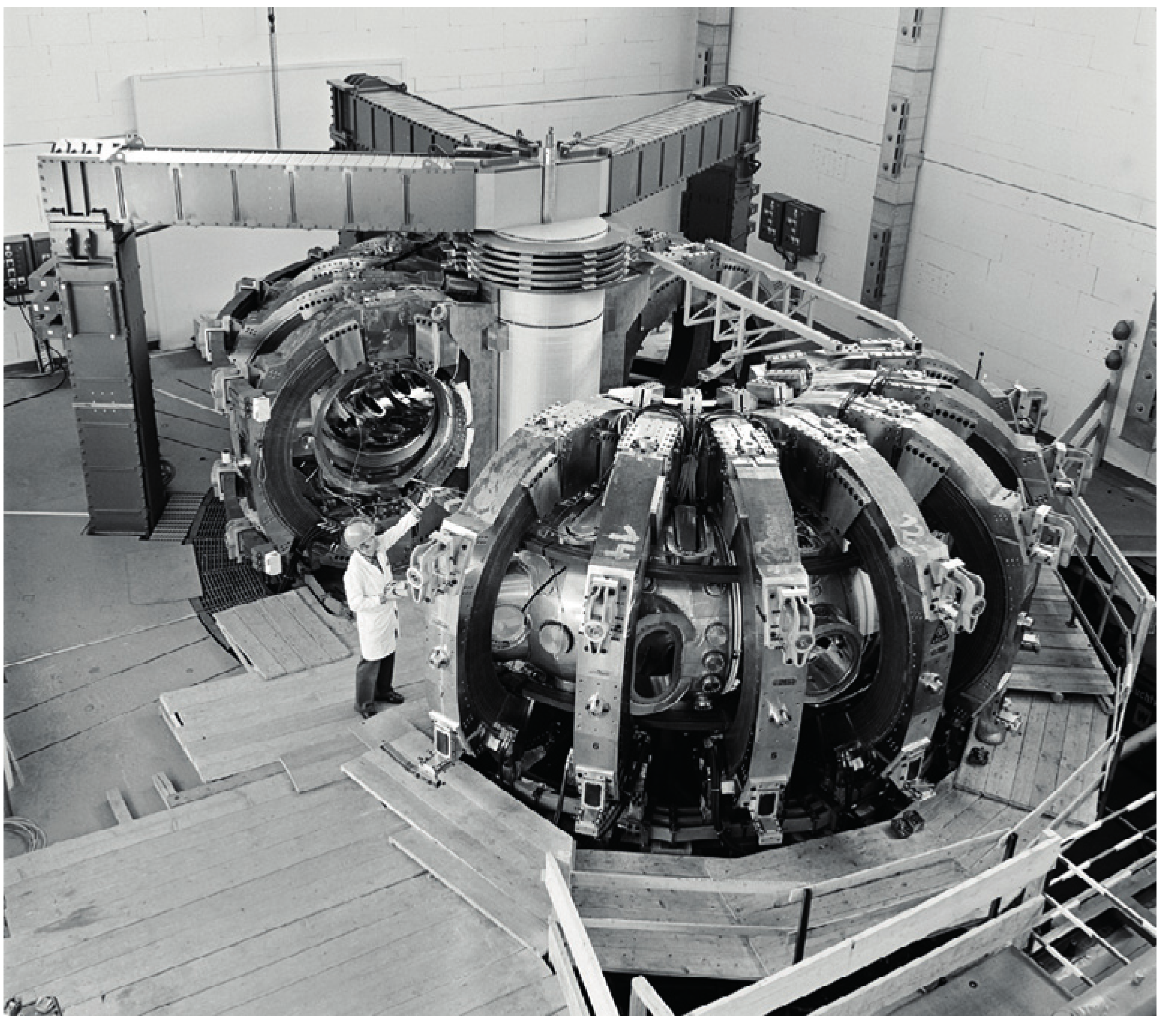

Figure 1.1: A dissasembled view of the Tokamak Experiment for Technology Orientated Research (TEXTOR) in Jülich, Germany from [1]

Much of this material is out of reach for anyone not already acquainted with the field, and while there are numerous textbooks on plasma physics in general ([4], [5], etc...), very few cover tokamak plasmas specifically or in enough detail as to satisfy an outside researcher seeking to understand their practical applications.

This thesis is an attempt to provide a solid mathematical foundation for the plasma physics of a tokamak. Beginning with idealized parameters, it presents the fundamental concepts and derivations needed to approach more advanced tokamak literature. This leaves room for alteration of assumptions made to suit the needs of other unique scenarios, filling in the gaps of other texts which disregard tokamak plasmas or the fundamentals of their behavior.

The main technique to be explored is the equilibrium boundary reconstruction of an ideal tokamak plasma. This problem is a foundational issue needed for almost every other task in a modern tokamak, and there are many algorithms used that approach this problem from different avenues. Here papers such as [6], [7], and others are explored and clarified to open up the field to those who may be interested.

Section 1.1: Intent

Because plasma is ionized, it carries a current and responds to magnetic forces. A tokamak leverages this effect with two sets of large magnetic coils oriented orthogonally around a toroidal vacuum vessel. By construction, one set of coils is meant to "drive" the plasma current toroidally, while another shapes and contains its poloidal profile within the chamber.

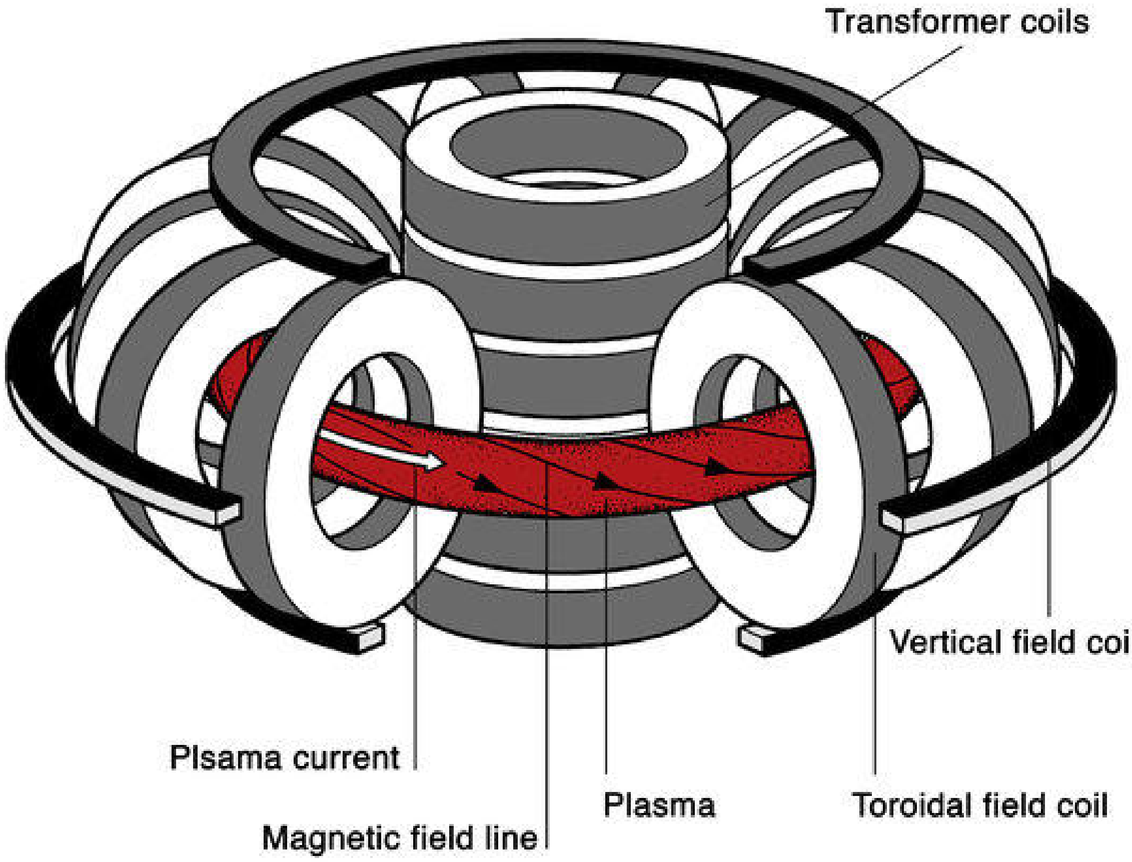

Figure 1.2: A diagram of the magnetic coils in a tokamak from [8]

This containment and shaping procedure is an extensive control problem, and the plasma's shape must be known *a priori*. Thus, to even effect the plasma in any meaningful way, the profile of its boundary must be first be determined, along with a litany of other small tasks.

By focusing on profile identification using known measurements from a given device, the breadth of this paper is restricted so that more attention can be given to mathematical details underlying the physics. While this task is a small part in the operation of a tokamak as a whole, it is nonetheless a fundamental part of the machine's operation, and provides a solid introduction to more sophisticated challenges.

Given a tokamak with specific device parameters and known measurements of a contained plasma, the goal is to approximate the plasma's corresponding boundary. This result allows for many other quantities to be computed, and is the basis for any control procedures that would follow.

The governing physics of this problem can be reduced to a partial differential equation (PDE) using some basic assumptions including axis-symmetry, negligible plasma resistance, and equilibrium conditions among others.

The PDE that results is a free-boundary equation however, so the domain for which it is applied is restricted to a region where it is homogeneous. Given measurements of the associated function at different points then allow for boundary conditions to be inferred, and the solution of the equation can be optimized to fit the measurements as closely as possible.

With the solution of the homogeneous equation in a restricted domain, a resulting fixed boundary can be approximated at the domain's edge so that other techniques can be applied in inhomogeneous regions. The homogeneous calculations are the main focus of this work, and inhomogeneous applications are left to further pursuits.

Section 1.2: Outline

The thesis is subdivided into the following parts:

First, Maxwell's equations are presented in the context of conducting fluids. These form the basis of understanding electromagnetic interactions. Assumptions are made to reduce these down to their most relevant forms, and variables of interest are identified and isolated. This gives the complete context of the problem in question, and allows for future adjustment based on different potential assumptions.

With the mathematical background in place, the resulting PDE is then discussed. A change of coordinates is given so that the result is both separable and conforms to the expected shape as closely as possible. Separation of variables is then used to give a solution that is well known in the literature; often presented, and almost never derived. Because the corresponding derivations are quite dense, much of the mathematical labor has been relegated to the appendices.

The method by which the resulting form is applied to known measurements is then discussed in more detail. This includes the needed information from the machine, as well as how to apply it practically. A few simple examples are given, and the generalized method is laid out.

In closing, an explanation is given of how to translate results from the homogeneous region into boundary conditions for the plasma in other regions. Common applications are briefly mentioned, and a technique to avoid inaccuracies near magnetic coils is presented. The direction of further work is then proposed.

CHAPTER 2: DERIVATIONS

Plasma is a conductive gas, and thus responds to both mechanical and electromagnetic forces. Maxwell's equations cover a basic understanding of continuous electrodynamic fields, while basic fluid dynamics can be incorporated to account for the medium in which they are contained.

Here equations are framed in terms of quantities more related to measurements as obtained from a tokamak, and general variables are replaced with others about which specific knowledge can be given.

Section 2.1: Maxwell's Equations

For completeness and accessibility, begin with Maxwell's equations, as they can be found in most general electrodynamics texts ([9], [10], etc...), and the theory behind them has been well developed and explored. The following electromagnetic quantities are thus presented in base SI units:

$\mathbf{D}$ is the **electric displacement** in Coulombs per square meter $(\mathrm{C} \cdot \mathrm{m}^{-2})$

$\mathbf{E}$ is the **electric field** in Volts per meter $(\mathrm{V} \cdot \mathrm{m}^{-1})$

τ is the **electric charge density** in Coulombs per cubic meter $(\mathrm{C} \cdot \mathrm{m}^{-3})$

$\mathbf{H}$ is the **magnetic field** in Amperes per meter $(\mathrm{A} \cdot \mathrm{m}^{-1})$

$\mathbf{B}$ is the **magnetic induction** in Teslas (T)

$\mathbf{J}$ is the **current density** in Amperes per square meter $(\mathrm{A} \cdot \mathrm{m}^{-2})$

Also included are the variables ϵ for **electric permittivity** $(\mathrm{C} \cdot \mathrm{V}^{-1} \cdot \mathrm{m}^{-1})$, and μ for **magnetic permeability** $(\mathrm{T} \cdot \mathrm{m} \cdot \mathrm{A}^{-1})$. These are tensors in general scenarios, but they reduce to constants in so called "linear materials." They come from the following definitions, which are taken as initial assumptions.

$$\mathbf{D} = \epsilon \mathbf{E}, \qquad \mathbf{B} = \mu \mathbf{H}$$

Because the permittivity and permeability are well known for a variety of materials, including a vacuum, it is more advantageous to write Maxwell's equations in terms of ϵ and μ than $\mathbf{D}$ and $\mathbf{H}$. By removing these unknown general variables, they are replaced with specific known quantities that can be applied in a variety of domains. The intent of the following work is of the same accord: to begin with general fundamentals and slowly remove extraneous values using more and more specific assumptions.

So, though complex scenarios may differ, the following assumptions apply in many cases and help to illustrate the development of further theory,

$$\frac{\partial}{\partial t}\epsilon = 0, \qquad \nabla\epsilon = 0, \qquad \nabla \times \mu^{-1} = 0 \tag{2.1}$$

With ϵ and μ then treated essentially as constants, **Maxwell's equations** can be presented, again in SI units, beginning with **Gauss's law**,

$$\begin{aligned}
\nabla \cdot \mathbf{D} &= \tau \\
\nabla \cdot (\epsilon \mathbf{E}) &= \tau \\
(\nabla \epsilon) \cdot \mathbf{E} + \epsilon \nabla \cdot \mathbf{E} &= \tau \\
\nabla \cdot \mathbf{E} &= \epsilon^{-1} \tau
\end{aligned} \tag{2.2}$$

then **Ampere's law**,

$$\begin{aligned}
\nabla \times \mathbf{H} - \frac{\partial}{\partial t}\mathbf{D} &= \mathbf{J} \\
\left(\nabla \times \mu^{-1}\right)\mathbf{B} + \mu^{-1}\nabla \times \mathbf{B} - \left(\frac{\partial}{\partial t}\epsilon\right)\mathbf{E} - \epsilon\frac{\partial}{\partial t}\mathbf{E} &= \mathbf{J} \\
\mu^{-1}\nabla \times \mathbf{B} - \epsilon\frac{\partial}{\partial t}\mathbf{E} &= \mathbf{J}
\end{aligned} \tag{2.3}$$

Faraday's law,

$$\nabla \times \mathbf{E} + \frac{\partial}{\partial t}\mathbf{B} = 0$$

and finally, the **magnetic monopole equation**,

$$\nabla \cdot \mathbf{B} = 0 \tag{2.4}$$

Boundary conditions can also be presented for the case of an interface between materials. They account for respective domains on either side, Ω_1 and Ω_2. Calling the boundary Γ, with a normal vector $\hat{\mathbf{n}}$ pointing from Ω_1 to Ω_2, any quantities associated with a particular domain are then labeled with respective subscripts.

Four conditions must be satisfied on Γ to account for surface charge densities τ_Γ and surface currents $\mathbf{J}_\Gamma$. First,

$$\hat{\mathbf{n}} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \tau_\Gamma \qquad\qquad \hat{\mathbf{n}} \times (\mathbf{E}_1 - \mathbf{E}_2) = 0$$

saying the transverse component of $\mathbf{E}$ does not change across a boundary, while the difference in the normal component of $\mathbf{D}$ is compensated by the surface charge, τ_Γ.

Similarly,

$$\hat{\mathbf{n}} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0 \qquad\qquad \hat{\mathbf{n}} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_\Gamma$$

where instead, the normal component of $\mathbf{B}$ does not change across a boundary, and the difference in the transverse component of $\mathbf{H}$ accounts for the surface current, $\mathbf{J}_\Gamma$.

Subsection 2.1.1: Ideal Conductive Fluid

Ohm's law is commonly thought of in the sense of circuits, but in a more general form, as listed in [10] and others, it can describe the velocity profile of a fluid as related to surrounding electromagnetic effects. Here κ represents the fluid's **conductivity** $(\mathrm{A} \cdot \mathrm{V}^{-1} \cdot \mathrm{m}^{-1})$, and

$$\mathbf{J} = \kappa(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Restructuring this equation in terms of $\mathbf{E}$ gives a way to replace effects of the electric field by their associated magnetic interactions in the fluid,

$$\mathbf{E} = \frac{\mathbf{J}}{\kappa} + \mathbf{B} \times \mathbf{v} \tag{2.5}$$

where hot plasma is often approximated as an ideal conductor so that $\kappa \to \infty$, giving a simplified form of 2.5,

$$\mathbf{E} = \mathbf{B} \times \mathbf{v} \tag{2.6}$$

Ampere's law (equation 2.3) can then be rewritten using the time derivative of 2.6, resulting in

$$\begin{aligned} \mu^{-1}\nabla \times \mathbf{B} - \epsilon\frac{\partial}{\partial t}\mathbf{E} &= \mathbf{J} \\ \mu^{-1}\nabla \times \mathbf{B} - \frac{\partial}{\partial t}(\mathbf{B} \times \mathbf{v}) &= \mathbf{J} \\ \mu^{-1}\nabla \times \mathbf{B} - \left(\frac{\partial}{\partial t}\mathbf{B}\right) \times \mathbf{v} - \mathbf{B} \times \frac{\partial}{\partial t}\mathbf{v} &= \mathbf{J} \end{aligned} \tag{2.7}$$

To fully simplify this representation, a few concessions must be made.

First, since electromagnetic effects take place on much shorter time scales than mechanical ones, a static velocity profile can be assumed in comparison, and second, by focusing on equilibrium scenarios alone, the magneto-static case can be assumed as well,

$$\frac{\partial}{\partial t}\mathbf{v} = 0, \quad \frac{\partial}{\partial t}\mathbf{B} = 0 \tag{2.8}$$

Ampere's law for a conductive fluid can then be written

$$\mu^{-1}\nabla \times \mathbf{B} = \mathbf{J} \tag{2.9}$$

and contains neither the electric field, nor the negligible effects of the fluid's velocity.

These assumptions also restructure Gauss' law and Faraday's law equations in terms of the electric field alone. Moreover, they allow for a solution of the electric field from known quantities (since they give both the divergence and the curl of $\mathbf{E}$, and ϵ and τ are known in a vacuum, which will be the domain of interest).

This renders the electric field an unneeded variable, and thus Ampere's law and the magnetic monopole equation become the relations of value.

Section 2.2: Cylindrical Symmetry

Employing a cylindrical coordinate system (r, ϕ, z), gives the ability to separate any vector field $\mathbf{C}$ into component fields labeled as follows:

$$\begin{aligned}\mathbf{C} &= C_r\hat{\boldsymbol{r}} + C_\phi\hat{\boldsymbol{\phi}} + C_z\hat{\boldsymbol{z}} \\ &= \mathbf{C}_r + \mathbf{C}_\phi + \mathbf{C}_z\end{aligned}$$

Since there is a symmetry associated with the toroidal ϕ variable in a tokamak, it is advisable to separate this part, $\mathbf{C}_\phi$, from the resulting decomposition. The remaining (r, z) components are then combined into a single variable $\mathbf{C}_\gamma$, referred to as the **poloidal part** of the vector field. Thus for any vector field there is a toroidal part and a poloidal part to be written

$$\mathbf{C}_\gamma = \mathbf{C} - \mathbf{C}_\phi, \qquad \mathbf{C}_\gamma = \mathbf{C}_r + \mathbf{C}_z, \qquad \mathbf{C} = \mathbf{C}_\phi + \mathbf{C}_\gamma$$

Section 2.3: Magnetic Potential

To combine Ampere's law and the magnetic monopole equation, they must be set in terms that allow them to relate directly. The following work accomplishes this task.

For any vector field $\mathbf{C}$ the identity $\nabla \cdot (\nabla \times \mathbf{C}) = 0$ always holds. Then the monopole equation (2.4) implies that $\mathbf{B}$ can be written in terms of the curl of some other vector field $\mathbf{A}$, called the **magnetic potential**.

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0 \\ \nabla \cdot \mathbf{B} &= \nabla \cdot (\nabla \times \mathbf{A}) \\ \mathbf{B} &= \nabla \times \mathbf{A} \end{aligned} \tag{2.10}$$

Since the curl of a gradient is zero, $\mathbf{A}$ contains a gauge symmetry where for any valid $\mathbf{A}_0$ and any scalar field a, there is an equally valid magnetic potential $\mathbf{A} = \mathbf{A}_0 + \nabla a$. The gauge symmetry doesn't come up in computations here, but is listed as a detail that may be important in other contexts.

Decomposing $\mathbf{A}$ and ∇ into cylindrical coordinates proceeds to variables that are more favorable to measurement,

$$\begin{aligned} \mathbf{B} &= \nabla \times \left(A_r \hat{\boldsymbol{r}} + A_\phi \hat{\boldsymbol{\phi}} + A_z \hat{\boldsymbol{z}} \right) \\ \mathbf{B} &= \left(r^{-1} \frac{\partial}{\partial \phi} A_z - \frac{\partial}{\partial z} A_\phi \right) \hat{\boldsymbol{r}} + \left(\frac{\partial}{\partial z} A_r - \frac{\partial}{\partial r} A_z \right) \hat{\boldsymbol{\phi}} + r^{-1} \left(\frac{\partial}{\partial r}(r A_\phi) - \frac{\partial}{\partial \phi} A_r \right) \hat{\boldsymbol{z}} \end{aligned}$$

where the toroidal symmetry about the z-axis results in

$$\frac{\partial}{\partial \phi} = 0 \tag{2.11}$$

implying

$$\mathbf{B} = -\frac{\partial}{\partial z} A_\phi \hat{\boldsymbol{r}} + \left(\frac{\partial}{\partial z} A_r - \frac{\partial}{\partial r} A_z \right) \hat{\boldsymbol{\phi}} + r^{-1} \frac{\partial}{\partial r}(r A_\phi) \hat{\boldsymbol{z}} \tag{2.12}$$

This can be written more succinctly by calling the **poloidal flux**

$$\psi = rA_\phi \tag{2.13}$$

and the **diamagnetic function**

$$T = rB_\phi = \left(\frac{\partial}{\partial z}A_r - \frac{\partial}{\partial r}A_z\right) \tag{2.14}$$

These quantities can be measured directly by specific probes on a tokamak, and are more easily relatable to the machine than either **B** or **A**.

Then

$$\frac{\partial}{\partial z}\psi = r\frac{\partial}{\partial z}A_\phi$$

and equation 2.12 is rewritten,

$$\mathbf{B} = r^{-1}\left(-\frac{\partial}{\partial z}\psi\hat{\boldsymbol{r}} + T\hat{\boldsymbol{\phi}} + \frac{\partial}{\partial r}\psi\hat{\boldsymbol{z}}\right)$$

where the components can be isolated,

$$\mathbf{B} = \mathbf{B}_\phi + \mathbf{B}_\gamma$$

$$\mathbf{B}_\gamma = r^{-1}\left(-\frac{\partial}{\partial z}\psi\hat{\boldsymbol{r}} + \frac{\partial}{\partial r}\psi\hat{\boldsymbol{z}}\right) \tag{2.15}$$

$$\mathbf{B}_\phi = r^{-1}T\hat{\boldsymbol{\phi}} \tag{2.16}$$

The axial symmetry 2.11 then reduces the gradient of ψ (and indeed, the gradient of any symmetrical scalar),

$$\nabla\psi = \frac{\partial}{\partial r}\psi\hat{\boldsymbol{r}} + r^{-1}\frac{\partial}{\partial\phi}\psi\hat{\boldsymbol{\phi}} + \frac{\partial}{\partial z}\psi\hat{\boldsymbol{z}}$$
$$\nabla\psi = \frac{\partial}{\partial r}\psi\hat{\boldsymbol{r}} + \frac{\partial}{\partial z}\psi\hat{\boldsymbol{z}}$$

so that with $\mathbf{B}_\gamma$ from equation 2.15,

$$\mathbf{B}\cdot\nabla\psi = \mathbf{B}_\gamma\cdot\nabla\psi = 0$$

showing that magnetic field lines are contained in surfaces of constant poloidal flux, ψ.

Moreover $\nabla\psi$ can be shown to be perpendicular to the plane formed by $\mathbf{B}_\gamma$ and $\hat{\boldsymbol{\phi}}$ since,

$$\nabla\psi\times\hat{\boldsymbol{\phi}} = -\frac{\partial}{\partial z}\psi\hat{\boldsymbol{r}} + \frac{\partial}{\partial r}\psi\hat{\boldsymbol{z}}$$
$$\nabla\psi\times\hat{\boldsymbol{\phi}} = r\mathbf{B}_\gamma \tag{2.17}$$

Thus $\mathbf{B}$ can be condensed again by combining equations 2.16 and 2.17,

$$\mathbf{B} = r^{-1}\left(T\hat{\boldsymbol{\phi}} + \nabla\psi\times\hat{\boldsymbol{\phi}}\right) \tag{2.18}$$

Because equilibrium conditions have been assumed, and current in open surfaces will dissipate, these surfaces of constant flux, ψ must form closed, nested shells in the plasma, and they become the basis of its topological structure (Figure 2.3).

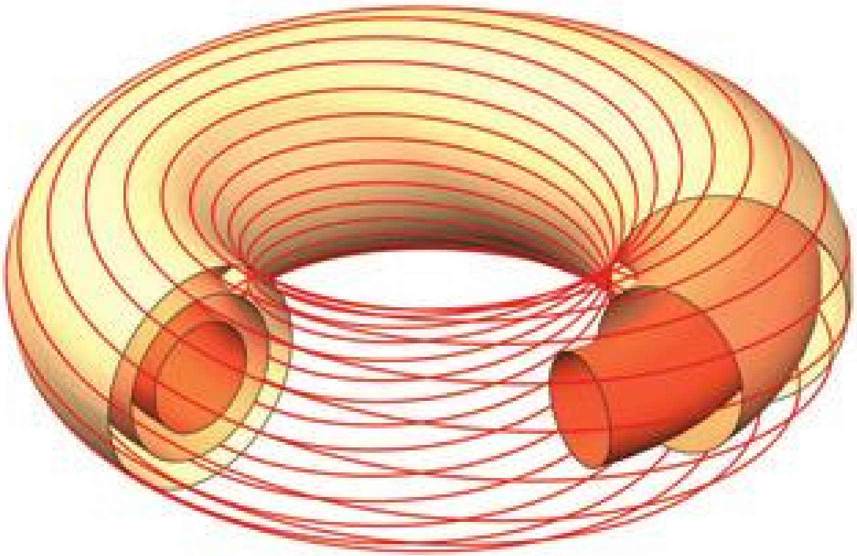

Figure 2.1: Flux shells in a tokamak plasma. Courtesy of [11]

Section 2.4: Incorporating Ampere's Law

The most important piece of this foundation comes from decomposing the current $\mathbf{J}$,

$$\begin{aligned}\mathbf{J} &= \mathbf{J}_r + \mathbf{J}_\phi + \mathbf{J}_z \\ &= \mathbf{J}_\phi + \mathbf{J}_\gamma\end{aligned}$$

where again, $\mathbf{J}_\gamma = \mathbf{J}_r + \mathbf{J}_z$.

Using the condensed Ampere's law, equation 2.9,

$$\mathbf{J} = \mu^{-1}\nabla \times \mathbf{B}$$

$$\mathbf{J} = \mu^{-1}\nabla \times \mathbf{B}_\gamma + \mu^{-1}\nabla \times \mathbf{B}_\phi$$

giving

$$\mathbf{J}_\phi = \mu^{-1}\nabla \times \mathbf{B}_\gamma \qquad\qquad \mathbf{J}_\gamma = \mu^{-1}\nabla \times \mathbf{B}_\phi$$

It follows that with the poloidal decomposition of **B** from equation 2.15, and the cylindrical definition of ∇,

$$\begin{aligned}
\mathbf{J}_\phi &= \mu^{-1}\nabla \times \mathbf{B}_\gamma \\
&= \mu^{-1}\nabla \times \left(-r^{-1}\frac{\partial}{\partial z}\psi\hat{\boldsymbol{r}} + r^{-1}\frac{\partial}{\partial r}\psi\hat{\boldsymbol{z}}\right) \\
&= \mu^{-1}\left(\frac{\partial}{\partial z}\left(-r^{-1}\frac{\partial}{\partial z}\psi\right) - \frac{\partial}{\partial r}\left(r^{-1}\frac{\partial}{\partial r}\psi\right)\right)\hat{\boldsymbol{\phi}} \\
\mathbf{J}_\phi &= -\mu^{-1}\left(\frac{\partial}{\partial r}\left(r^{-1}\frac{\partial}{\partial r}\psi\right) + \frac{\partial}{\partial z}\left(r^{-1}\frac{\partial}{\partial z}\psi\right)\right)\hat{\boldsymbol{\phi}} \\
J_\phi &= -\mu^{-1}\Delta^*\psi
\end{aligned} \tag{2.19}$$

where Δ^* is defined as the elliptic operator,

$$\Delta^* = \nabla \cdot \left(r^{-1}\nabla\right) = \frac{\partial}{\partial r}\left(r^{-1}\frac{\partial}{\partial r}\right) + \frac{\partial}{\partial z}\left(r^{-1}\frac{\partial}{\partial z}\right) \tag{2.20}$$

and is refferred to as the **modified Laplacian**.

Equation 2.19 is the starting point in finding the poloidal boundary of ψ and thus the boundary of the plasma as well. This is because the current in a vacuum is null, the plasma is surrounded entirely by vacuum, and thus ψ can be found in the region surrounding the plasma and up to its edge using the resulting homogeneous form. Then, with a known ψ, the remaining quantities such as the magnetic field and electric field follow naturally.

The rest of this material is covered to show how to find the full representation of $\mathbf{J}$ once appropriate boundary conditions have been set up and ψ computed.

Consider the poloidal component of the current, equation 2.16, also decomposed into cylindrical coordinates,

$$
\begin{aligned}
\mathbf{J}_\gamma &= \mu^{-1}\nabla \times \mathbf{B}_\phi \\
&= \mu^{-1}\nabla \times \left(B_\phi \hat{\boldsymbol{\phi}}\right) \\
&= \mu^{-1}\left(B_\phi \nabla \times \hat{\boldsymbol{\phi}} + \nabla B_\phi \times \hat{\boldsymbol{\phi}}\right) \\
&= \mu^{-1}\left(\frac{B_\phi}{r}\hat{\boldsymbol{z}} + \nabla B_\phi \times \hat{\boldsymbol{\phi}}\right) \\
&= (\mu r)^{-1}\left(B_\phi \hat{\boldsymbol{z}} + r\nabla B_\phi \times \hat{\boldsymbol{\phi}}\right) \\
&= (\mu r)^{-1}\left(B_\phi \hat{\boldsymbol{z}} + (\nabla(rB_\phi) - B_\phi \nabla r) \times \hat{\boldsymbol{\phi}}\right) \\
&= (\mu r)^{-1}\left(B_\phi \hat{\boldsymbol{z}} + \nabla(rB_\phi) \times \hat{\boldsymbol{\phi}} - B_\phi \hat{\boldsymbol{r}} \times \hat{\boldsymbol{\phi}}\right) \\
&= (\mu r)^{-1}\nabla(rB_\phi) \times \hat{\boldsymbol{\phi}} \\
\mathbf{J}_\gamma &= (\mu r)^{-1}\nabla T \times \hat{\boldsymbol{\phi}}
\end{aligned}
\tag{2.21}
$$

$\mathbf{J}$ is then written in total by combining 2.19 and 2.21,

$$
\begin{aligned}
\mathbf{J} &= \mathbf{J}_\phi + \mathbf{J}_\gamma \\
\mathbf{J} &= -\mu^{-1}\Delta^*\psi \hat{\boldsymbol{\phi}} + \frac{\nabla T}{\mu r} \times \hat{\boldsymbol{\phi}}
\end{aligned}
\tag{2.22}
$$

Section 2.5: Isotropic Pressure

A complete representation of the plasma's profile then requires two more assumptions:

First, approximate the plasma as having uniform permittivity and permeability (ϵ and μ constant throughout), and second, consider the **fluid pressure** P of the plasma to be balanced with the Lorentz forces from the magnetic field confining it; ie.,

$$\nabla P = \mathbf{J} \times \mathbf{B} \tag{2.23}$$

It follows immediately that

$$\nabla P \cdot \mathbf{B} = 0, \qquad\qquad \nabla P \cdot \mathbf{J} = 0$$

showing that P is constant on each magnetic surface, and that the current $\mathbf{J}$ is as well. These magnetic surfaces coincide with the flux surfaces from earlier.

Thus P can be written exclusively in terms of ψ so that $P(\psi)$ is well defined, and moreover, the magnetic and current surfaces have the same profile (albeit with different values and perpendicular in orientation).

Then since P is dependent only on ψ, and ψ depends only on (r, z), P is also dependent only on the poloidal coordinates, and is axis-symmetric. Then substituting equation 2.21 for $\mathbf{J}$ in addition to these observations,

$$\begin{aligned} \nabla P \cdot \mathbf{J} &= 0 \\ \nabla P \cdot \mathbf{J}_\gamma &= 0 \\ \nabla P \cdot (\nabla T \times \hat{\boldsymbol{\phi}}) &= 0 \end{aligned}$$

showing ∇P and ∇T are collinear, $\nabla\psi$ and ∇T are collinear, and again, $T(\psi)$ is well defined.

With $\mathbf{B}$ from equation 2.18 and $\mathbf{J}$ from equation 2.22, the pressure balance formula 2.23 is transformed,

$$\nabla P = \mathbf{J} \times \mathbf{B}$$

$$\nabla P = (\mathbf{J}_\gamma + \mathbf{J}_\phi) \times (\mathbf{B}_\gamma + \mathbf{B}_\phi)$$

$$\nabla P = \mathbf{J}_\gamma \times \mathbf{B}_\gamma + \mathbf{J}_\gamma \times \mathbf{B}_\phi + \mathbf{J}_\phi \times \mathbf{B}_\gamma + \mathbf{J}_\phi \times \mathbf{B}_\phi$$

$$\nabla P = \mathbf{J}_\gamma \times \mathbf{B}_\phi + \mathbf{J}_\phi \times \mathbf{B}_\gamma$$

$$\nabla P = \left(\frac{\nabla T \times \hat{\boldsymbol{\phi}}}{\mu r}\right) \times \left(\frac{T}{r}\hat{\boldsymbol{\phi}}\right) + \left(\frac{-\Delta^* \psi}{\mu}\hat{\boldsymbol{\phi}}\right) \times \left(\frac{\nabla \psi}{r} \times \hat{\boldsymbol{\phi}}\right)$$

$$\nabla P = \left(\frac{\nabla T \times \hat{\boldsymbol{\phi}}}{\mu r}\right) \times \left(\frac{T}{r}\hat{\boldsymbol{\phi}}\right) + \left(\frac{\nabla \psi}{r} \times \hat{\boldsymbol{\phi}}\right) \times \left(\frac{\Delta^* \psi}{\mu}\hat{\boldsymbol{\phi}}\right)$$

$$\nabla P = (\mu r)^{-1} \left(\frac{T}{r}(\nabla T \times \hat{\boldsymbol{\phi}}) \times \hat{\boldsymbol{\phi}} + \Delta^* \psi (\nabla \psi \times \hat{\boldsymbol{\phi}}) \times \hat{\boldsymbol{\phi}}\right)$$

Standard vector identities are then applied,

$$(\nabla \psi \times \hat{\boldsymbol{\phi}}) \times \hat{\boldsymbol{\phi}} = -\nabla \psi \qquad (\nabla T \times \hat{\boldsymbol{\phi}}) \times \hat{\boldsymbol{\phi}} = -\nabla T$$

and this results in

$$\nabla P = -(\mu r)^{-1} \left(\frac{T}{r}\nabla T \ + \Delta^* \psi \nabla \psi\right)$$

Since $T(\psi)$ and $P(\psi)$ are well defined, they can be written out as

$$\nabla T(\psi) = T'(\psi)\nabla\psi, \qquad \nabla P(\psi) = P'(\psi)\nabla\psi$$

so that substitution produces

$$
\begin{aligned}
P'(\psi)\nabla\psi &= -(\mu r)^{-1}\left(\frac{T(\psi)}{r}T'(\psi)\nabla\psi + \Delta^*\psi\nabla\psi\right) \\
-\Delta^*\psi &= \mu r P'(\psi) + \frac{T(\psi)}{r}T'(\psi)
\end{aligned}
\tag{2.24}
$$

which is known as the **Grad-Shafranov equation**.

This equation depends only on ψ, and can be solved to find P, T, and thus a complete profile of the plasma's current from equation 2.22.

CHAPTER 3: SEPARATION OF VARIABLES

Consider now the modified Laplacian,

$$\Delta^* = \nabla \cdot \left(\frac{1}{r}\nabla\right) = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial z}\right) \tag{2.20 Revisited}$$

Finding the boundary of the plasma in a tokamak is intimately linked with finding a solution to the equation involving this operator, namely,

$$-\mu J_\phi = \Delta^* \psi \tag{2.19 Revisited}$$

The forcing function J_ϕ imposes a hardship, but since plasmas have an extremely strong re-neutralizing force ([4],[5]), it is a realistic assumption to say that no current will "leak" off the plasma.

Thus, in the vacuum region surrounding the plasma, $J_\phi = 0$, reducing the equation to a homogeneous, or "Laplacian" style, problem,

$$\begin{aligned} 0 &= \Delta^* \psi \\ 0 &= \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\psi\right) + \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial z}\psi\right) \end{aligned}$$

Equations of this form are often solved by separation of variables. This is a method by which the two-variable function of $\psi(r, z)$ is assumed to be a product of single-variable functions.

This allows the partial differential equation to be broken into a sum of single-variable, ordinary equations that must be constant to retain the appropriate relationship under variation of the coordinates.

This is not always possible, and the equation may often require a leading function or a change of coordinates to put it into a form for which this technique can be applied (See [12]). For instance, this equation is not separable in Cartesian coordinates, and requires a more involved power series expansion for its solution.

In addition, using a coordinate system that closely matches the topology of the solution will remove much of the error that can arise in practical applications. For these reasons, a toroidal coordinate system is applied, and because vector components are not used and derivatives here become quite dense, subscript notation will instead denote partial derivatives for clarity and space.

That is, from here on out, for any function f and variable x,

$$\frac{\partial}{\partial x} f = \partial_x f = f_x \tag{3.1}$$

Section 3.1: Toroidal Coordinates

As described in [13], toroidal coordinates identify points in the right half plane using a relationship between two focal points placed at equivalent distances about the axis of symmetry, and the orthogonal intersection of "Apollonian circles" involving these foci; toroidal symmetry implies each half plane is a mirror of the other.

Constant values of the coordinates produce these circles, and the only points not defined uniquely are the foci themselves.

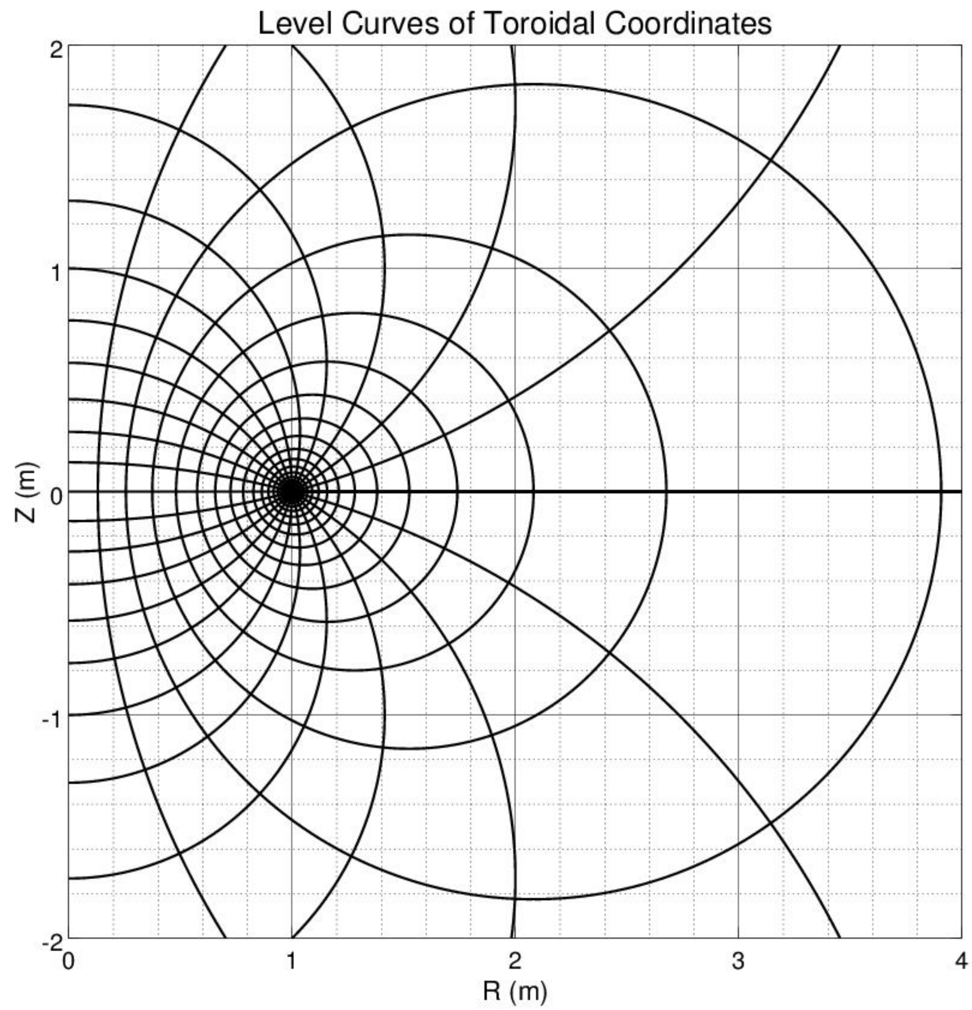

Figure 3.1: Constant values of the toroidal coordinates in the right half plane with a focus at $(r = 1, z = 0)$.

Care must be taken to include the foci well within the expected boundary of the plasma, and the centroid of the machine itself is a valid starting place for this.

The goal is then to rewrite the augmented Laplacian,

$$\Delta^* = \partial_r \left(r^{-1}\partial_r\right) + \partial_z \left(r^{-1}\partial_z\right) \tag{2.20 Revisited}$$

in toroidal coordinates, so that the resulting PDE is both separable and its coordinates conform to the expected solution as closely as possible. Extended derivations are given in appendix A.

To describe the coordinates more rigorously, consider some given foci $(\pm r_0, z_0)$ along with an arbitrary, non-focal point using cylindrical coordinates (r, z). Unless otherwise noted, points are always in the right half plane.

The corresponding toroidal coordinates for the arbitrary point, (ζ, η), are given by first computing the corresponding distances to the foci,

$$d_- = \sqrt{(r + r_0)^2 + (z - z_0)^2} \qquad\qquad d_+ = \sqrt{(r - r_0)^2 + (z - z_0)^2}$$

so that the coordinates are then,

$$\zeta = \ln \frac{d_-}{d_+} \qquad\qquad \eta = \arccos\left(\frac{d_-^2 + d_+^2 - 4r_0^2}{2d_- d_+}\right) \tag{3.2}$$

where $\eta < \pi$ for $z > z_0$, and $\eta > \pi$ for $z < z_0$.

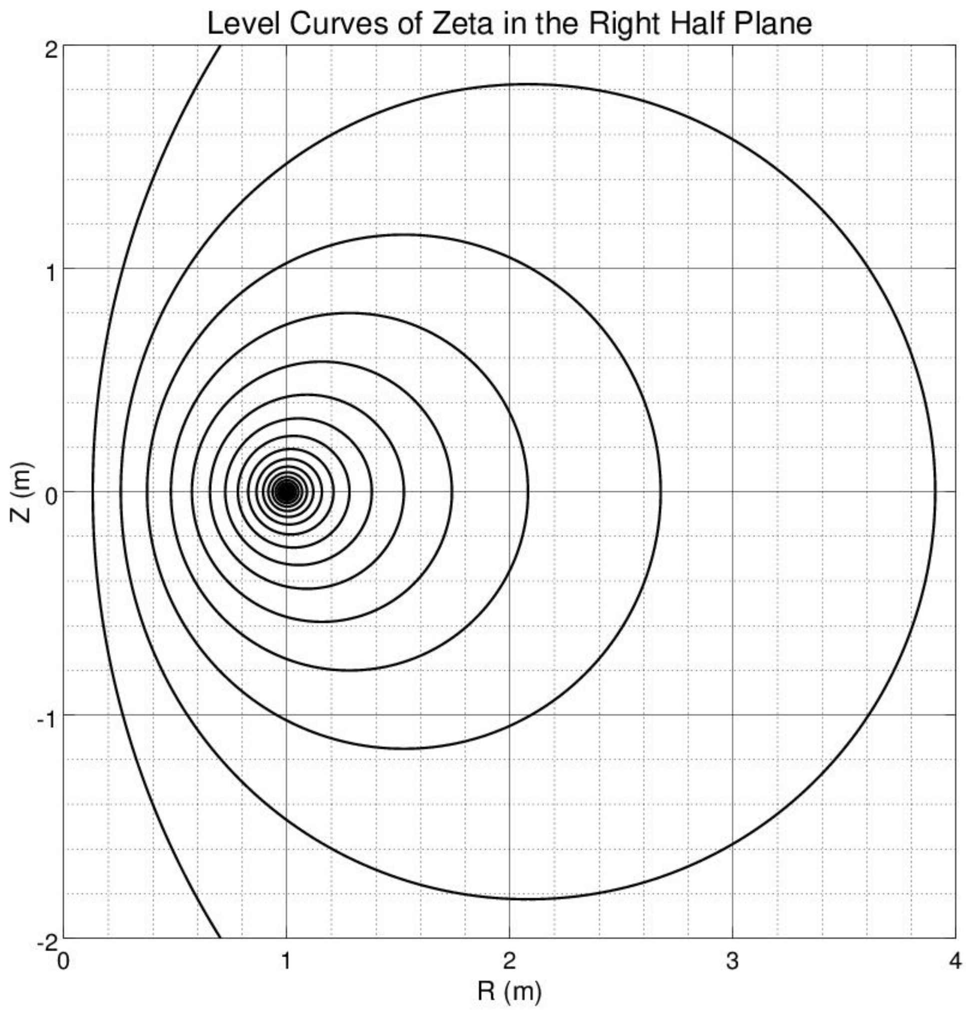

Figure 3.2: Constant values of ζ in the right half plane with a focus at $(r = 1, z = 0)$.

Constant values of ζ and η produce circles either surrounding the poles, as in the case of ζ (Figure 3.2), or intersecting them, as in the case of η (Figure 3.3).

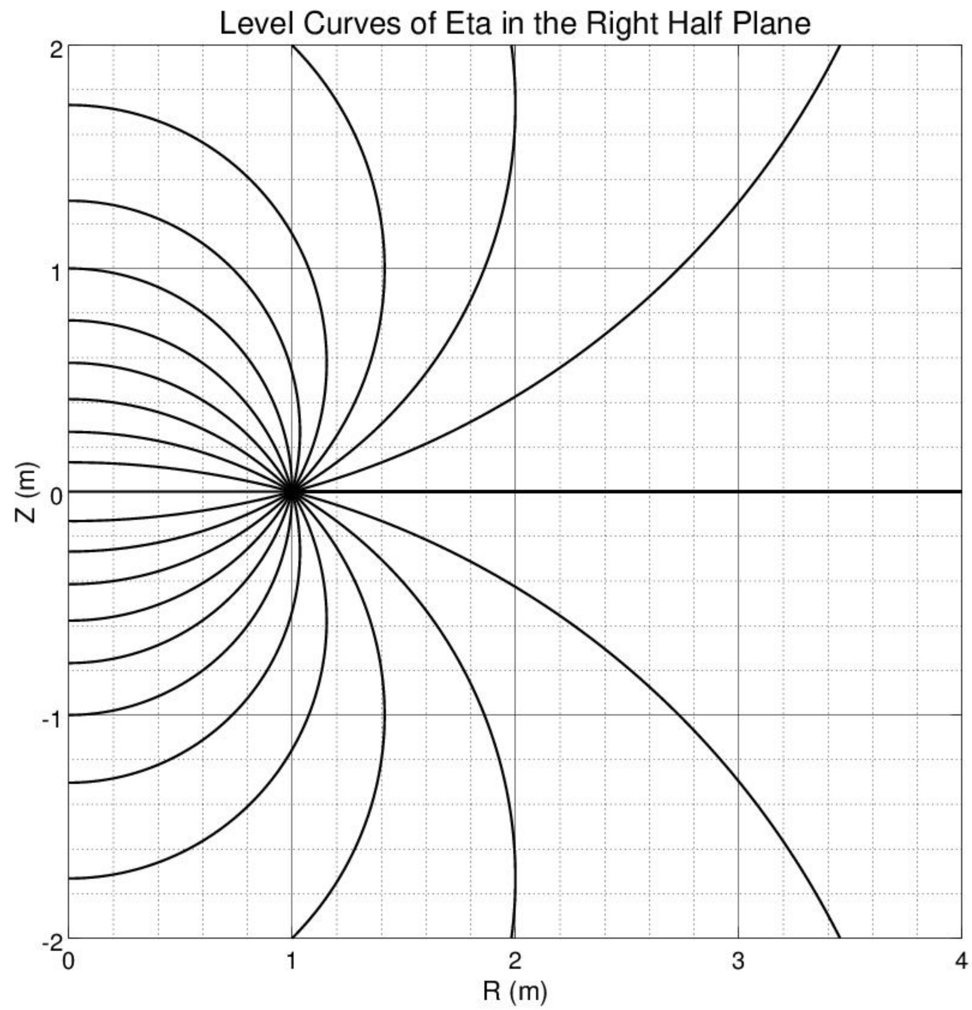

Figure 3.3: Constant values of η in the right half plane with a focus at $(r = 1, z = 0)$.

Inverse transformations can also be given,

$$r = \frac{r_0 \sinh \zeta}{\cosh \zeta - \cos \eta}, \qquad z = z_0 + \frac{r_0 \sin \eta}{\cosh \zeta - \cos \eta} \tag{3.3}$$

and are used to frame d_- and d_+ in hyperbolic terms,

$$d_- = r_0\sqrt{2\frac{\cosh\zeta+\sinh\zeta}{\cosh\zeta-\cos\eta}}, \qquad d_+ = r_0\sqrt{2\frac{\cosh\zeta-\sinh\zeta}{\cosh\zeta-\cos\eta}}$$

where the derivations are again given explicitly in appendix A.

For the partial derivatives in hyperbolic terms, it is convenient to notice that

$$\begin{aligned} r-r_0 &= r_0\left(\frac{\cos\eta}{\cosh\zeta-\cos\eta}-\frac{\cosh\zeta-\sinh\zeta}{\cosh\zeta-\cos\eta}\right)\\ r+r_0 &= r_0\left(\frac{\cosh\zeta+\sinh\zeta}{\cosh\zeta-\cos\eta}-\frac{\cos\eta}{\cosh\zeta-\cos\eta}\right)\end{aligned}$$

and

$$d_-^2 = 2r_0^2\left(\frac{\cosh\zeta+\sinh\zeta}{\cosh\zeta-\cos\eta}\right), \qquad d_+^2 = 2r_0^2\left(\frac{\cosh\zeta-\sinh\zeta}{\cosh\zeta-\cos\eta}\right)$$

$$d_-^2+d_+^2 = \frac{4r_0^2\cosh\zeta}{\cosh\zeta-\cos\eta}, \qquad d_-^2-d_+^2 = \frac{4r_0^2\sinh\zeta}{\cosh\zeta-\cos\eta} = 4rr_0$$

$$d_-d_+ = 2r_0^2\sqrt{\left(\frac{\cosh\zeta+\sinh\zeta}{\cosh\zeta-\cos\eta}\right)\left(\frac{\cosh\zeta-\sinh\zeta}{\cosh\zeta-\cos\eta}\right)} = \frac{2r_0^2}{\cosh\zeta-\cos\eta}$$

Moreover, the partial derivatives of the components also become relevant,

$$\begin{aligned} \partial_r d_- &= \frac{r+r_0}{d_-}, & \partial_r d_+ &= \frac{r-r_0}{d_+}\\ \partial_z d_- &= \frac{z-z_0}{d_-}, & \partial_z d_+ &= \frac{z-z_0}{d_+}\end{aligned}$$

With these basic formulas established, the first partials of the coordinates can be solved and used for the transformation of the operator. Yet again, these calculations are quite dense, and interested readers may refer to appendix A for the full computations.

$$\zeta_r = \partial_r \ln \frac{d_-}{d_+} = \frac{1 - \cosh\zeta\cos\eta}{r_0}, \qquad \zeta_z = \partial_z \ln \frac{d_-}{d_+} = \frac{-\sinh\zeta\sin\eta}{r_0}$$

$$\begin{aligned}
\eta_r &= \partial_r \arccos\left(\frac{d_-^2 + d_+^2 - 4r_0^2}{2d_-d_+}\right) = \frac{-\sinh\zeta\sin\eta}{r_0} = \zeta_z \\
\eta_z &= \partial_z \arccos\left(\frac{d_-^2 + d_+^2 - 4r_0^2}{2d_-d_+}\right) = \frac{\cosh\zeta\cos\eta - 1}{r_0} = -\zeta_r
\end{aligned}$$

Then, collected with their associated partials,

$$\begin{aligned}
\zeta_r &= -\eta_z = \frac{1 - \cosh\zeta\cos\eta}{r_0}, & \zeta_z &= \eta_r = \frac{-\sinh\zeta\sin\eta}{r_0} \\
\partial_\zeta\zeta_r &= -\partial_\zeta\eta_z = \frac{-\sinh\zeta\cos\eta}{r_0}, & \partial_\zeta\zeta_z &= \partial_\zeta\eta_r = \frac{-\cosh\zeta\sin\eta}{r_0} \\
\partial_\eta\zeta_r &= -\partial_\eta\eta_z = \frac{\cosh\zeta\sin\eta}{r_0}, & \partial_\eta\zeta_z &= \partial_\eta\eta_r = \frac{-\sinh\zeta\cos\eta}{r_0}
\end{aligned} \tag{3.4}$$

so that some useful identities emerge,

$$\begin{aligned}
\partial_\zeta\zeta_r &= \partial_\eta\zeta_z, & \partial_\eta\zeta_r + \partial_\zeta\zeta_z &= 0 \\
\partial_\zeta\eta_z + \partial_\eta\eta_r &= 0, & \partial_\eta\eta_z &= \partial_\zeta\eta_r
\end{aligned} \tag{3.5}$$

These are the main preparations, and with them it is possible to undertake the actual transformation after expanding the form of the operator,

$$\begin{aligned}
\Delta^* &= \partial_r(r^{-1}\partial_r) + \partial_z(r^{-1}\partial_z) \\
&= r^{-1}\left(\partial_r^2 + \partial_z^2 - r^{-1}\partial_r\right)
\end{aligned}$$

By using the identities from 3.4, the partials in cylindrical coordinates can be written in terms of just one of the toroidal variables. Repeated applications of the product and chain rules then give,

$$\partial_r = \zeta_r \partial_\zeta + \eta_r \partial_\eta = \zeta_r \partial_\zeta + \zeta_z \partial_\eta$$

$$\begin{aligned}
\partial_r^2 &= (\zeta_r \partial_\zeta + \zeta_z \partial_\eta)(\zeta_r \partial_\zeta + \zeta_z \partial_\eta) \\
&= (\zeta_r \partial_\zeta + \zeta_z \partial_\eta)(\zeta_r \partial_\zeta) \\
&\quad + (\zeta_r \partial_\zeta + \zeta_z \partial_\eta)(\zeta_z \partial_\eta) \\
&= (\zeta_r \partial_\zeta \zeta_r + \zeta_z \partial_\eta \zeta_r)\partial_\zeta + \zeta_r(\zeta_r \partial_\zeta + \zeta_z \partial_\eta)\partial_\zeta \\
&\quad + (\zeta_r \partial_\zeta \zeta_z + \zeta_z \partial_\eta \zeta_z)\partial_\eta + \zeta_z(\zeta_r \partial_\zeta + \zeta_z \partial_\eta)\partial_\eta
\end{aligned}$$

$$\partial_z = \zeta_z \partial_\zeta + \eta_z \partial_\eta = \zeta_z \partial_\zeta - \zeta_r \partial_\eta$$

$$\begin{aligned}
\partial_z^2 &= (\zeta_z \partial_\zeta - \zeta_r \partial_\eta)(\zeta_z \partial_\zeta - \zeta_r \partial_\eta) \\
&= (\zeta_z \partial_\zeta - \zeta_r \partial_\eta)(\zeta_z \partial_\zeta) \\
&\quad + (\zeta_r \partial_\eta - \zeta_z \partial_\zeta)(\zeta_r \partial_\eta) \\
&= (\zeta_z \partial_\zeta \zeta_z - \zeta_r \partial_\eta \zeta_z)\partial_\zeta + \zeta_z(\zeta_z \partial_\zeta - \zeta_r \partial_\eta)\partial_\zeta \\
&\quad + (\zeta_r \partial_\eta \zeta_r - \zeta_z \partial_\zeta \zeta_r)\partial_\eta + \zeta_r(\zeta_r \partial_\eta - \zeta_z \partial_\zeta)\partial_\eta
\end{aligned}$$

Combining the results and applying the remaining identities from 3.5 show that

$$\begin{aligned}
\partial_r^2 + \partial_z^2 &= \left[\zeta_r(\partial_\zeta \zeta_r - \partial_\eta \zeta_z) + \zeta_z(\partial_\eta \zeta_r + \partial_\zeta \zeta_z)\right]\partial_\zeta \\
&\quad + \left[\zeta_r(\zeta_r \partial_\zeta + \zeta_z \partial_\eta) + \zeta_z(\zeta_z \partial_\zeta - \zeta_r \partial_\eta)\right]\partial_\zeta \\
&\quad + \left[\zeta_z(\partial_\eta \zeta_z - \partial_\zeta \zeta_r) + \zeta_r(\partial_\zeta \zeta_z + \partial_\eta \zeta_r)\right]\partial_\eta \\
&\quad + \left[\zeta_z(\zeta_r \partial_\zeta + \zeta_z \partial_\eta) + \zeta_r(\zeta_r \partial_\eta - \zeta_z \partial_\zeta)\right]\partial_\eta \\
&= \left[\zeta_r(\zeta_r \partial_\zeta + \zeta_z \partial_\eta) + \zeta_z(\zeta_z \partial_\zeta - \zeta_r \partial_\eta)\right]\partial_\zeta \\
&\quad + \left[\zeta_z(\zeta_r \partial_\zeta + \zeta_z \partial_\eta) + \zeta_r(\zeta_r \partial_\eta - \zeta_z \partial_\zeta)\right]\partial_\eta \\
&= (\zeta_r^2 + \zeta_z^2)\partial_\zeta^2 + (\zeta_r^2 + \zeta_z^2)\partial_\eta^2
\end{aligned}$$

where

$$
\begin{aligned}
\zeta_r^2 + \zeta_z^2 &= \frac{(1-\cosh\zeta\cos\eta)^2}{r_0^2} + \frac{\sinh^2\zeta\sin^2\eta}{r_0^2} \\
&= r_0^{-2}\left((1-\cosh\zeta\cos\eta)^2 + \sinh^2\zeta\sin^2\eta\right) \\
&= r_0^{-2}\left(1+\cosh^2\zeta\cos^2\eta + \sinh^2\zeta\sin^2\eta - 2\cosh\zeta\cos\eta\right) \\
&= r_0^{-2}\left(\cos^2\eta + \cosh^2\zeta\cos^2\eta + (1+\sinh^2\zeta)\sin^2\eta - 2\cosh\zeta\cos\eta\right) \\
&= r_0^{-2}\left(\cos^2\eta + \cosh^2\zeta - 2\cosh\zeta\cos\eta\right) = \frac{(\cosh\zeta-\cos\eta)^2}{r_0^2}
\end{aligned}
$$

and

$$
\begin{aligned}
r^{-1}\partial_r &= r^{-1}\zeta_r\partial_\zeta + r^{-1}\eta_r\partial_\eta = r^{-1}\zeta_r\partial_\zeta + r^{-1}\zeta_z\partial_\eta \\
&= \frac{\cosh\zeta-\cos\eta}{r_0\sinh\zeta}\left(\frac{1-\cosh\zeta\cos\eta}{r_0}\right)\partial_\zeta + \frac{\cosh\zeta-\cos\eta}{r_0\sinh\zeta}\left(\frac{-\sinh\zeta\sin\eta}{r_0}\right)\partial_\eta \\
&= \frac{\cosh\zeta-\cos\eta}{r_0^2}\left(\frac{1-\cosh\zeta\cos\eta}{\sinh\zeta}\partial_\zeta - (\sin\eta)\partial_\eta\right)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\partial_r^2 + \partial_z^2 - r^{-1}\partial_r = \frac{\cosh\zeta-\cos\eta}{r_0^2}&\left[\left(\frac{\cosh\zeta\cos\eta-1}{\sinh\zeta}\right)\partial_\zeta + (\cosh\zeta-\cos\eta)\,\partial_\zeta^2\right. \\
&\quad \left. +(\sin\eta)\partial_\eta + (\cosh\zeta-\cos\eta)\partial_\eta^2\right] \\
= \frac{\cosh\zeta-\cos\eta}{r_0^2}&\left[\sinh\zeta\left(\left(\frac{\cosh\zeta\cos\eta-1}{\sinh^2\zeta}\right)\partial_\zeta + \frac{\cosh\zeta-\cos\eta}{\sinh\zeta}\partial_\zeta^2\right)\right. \\
&\quad \left. +(\sin\eta)\partial_\eta + (\cosh\zeta-\cos\eta)\partial_\eta^2\right] \\
= \frac{\cosh\zeta-\cos\eta}{r_0^2}&\left[(\sinh\zeta)\partial_\zeta\left(\frac{\cosh\zeta-\cos\eta}{\sinh\zeta}\partial_\zeta\right) + \partial_\eta\left((\cosh\zeta-\cos\eta)\partial_\eta\right)\right]
\end{aligned}
$$

all together implying that the modified Laplacian, $\Delta^* = r^{-1}\left(\partial_r^2 + \partial_z^2 - r^{-1}\partial_r\right)$, is written in toroidal coordinates as,

$$\Delta^* = \frac{(\cosh\zeta - \cos\eta)^2}{r_0^3}\left[\partial_\zeta\left(\frac{\cosh\zeta - \cos\eta}{\sinh\zeta}\partial_\zeta\right) + \partial_\eta\left(\frac{\cosh\zeta - \cos\eta}{\sinh\zeta}\partial_\eta\right)\right] \tag{3.6}$$

Section 3.2: Separation of Variables

This augmented Laplacian from 3.6 can then applied to the poloidal flux, ψ, as given in equation 2.19.

Assuming that $\psi(\zeta, \eta)$ can be represented as the product of single-variable functions leads to a representation of $\Delta^*\psi$ as a sum of single-variable functions. To preserve the homogeneity with variations in each coordinate, each summand must then be constant, where the total sum is null.

To be explicit, assume $\psi(\zeta, \eta)$ can be represented as the product of single-variable functions $f(\zeta)$ and $g(\eta)$, and introduce a leading function $R(\zeta, \eta)$ to account for the embedded terms of the equation,

$$\psi(\zeta, \eta) = R(\zeta, \eta) f(\zeta) g(\eta) \tag{3.7}$$

With this replacement,

$$\begin{aligned}\Delta^*\psi &= \frac{(\cosh\zeta - \cos\eta)^2}{r_0^3}\left[\partial_\zeta\left(\frac{\cosh\zeta - \cos\eta}{\sinh\zeta}\partial_\zeta\psi\right) + \partial_\eta\left(\frac{\cosh\zeta - \cos\eta}{\sinh\zeta}\partial_\eta\psi\right)\right] \\ &= \frac{(\cosh\zeta - \cos\eta)^2}{r_0^3}\left[g\partial_\zeta\left(\frac{\cosh\zeta - \cos\eta}{\sinh\zeta}\partial_\zeta(Rf)\right) + f\partial_\eta\left(\frac{\cosh\zeta - \cos\eta}{\sinh\zeta}\partial_\eta(Rg)\right)\right]\end{aligned} \tag{3.8}$$

The summands are of the same form, so abuse notation by calling

$$A = \frac{\cosh\zeta - \cos\eta}{\sinh\zeta} \tag{3.9}$$

for now, so that

$$\Delta^*\psi = \frac{(\cosh\zeta - \cos\eta)^2}{r_0^3}\left[g\partial_\zeta\left(A\partial_\zeta(Rf)\right) + f\partial_\eta\left(A\partial_\eta(Rg)\right)\right]$$

A general expansion can then be found for the common term, and the results used in the context of each partial.

To this end, let an arbitrary function h (of the same single variable as the associated partial) stand in for f and g respectively. That is to say, take

$$\begin{aligned}
\partial(A\partial(Rh)) &= \partial A\partial(Rh) + A\partial^2(Rh) \\
&= \partial A(h\partial R + h'R) + A(h\partial^2 R + 2h'\partial R + h''R) \\
&= h(\partial A\partial R + A\partial^2 R) + h'(R\partial A + 2A\partial R) + h''AR
\end{aligned}$$

so that dividing through by ARh produces,

$$\frac{\partial(A\partial(Rh))}{ARh} = \frac{\partial A\partial R}{AR} + \frac{\partial^2 R}{R} + \frac{h'}{h}\left(\frac{\partial A}{A} + 2\frac{\partial R}{R}\right) + \frac{h''}{h}$$

Written in these terms equation 3.8 becomes,

$$\Delta^*\psi = \frac{(\cosh\zeta - \cos\eta)^2}{r_0^3}ARfg\left[\frac{\partial_\zeta\left(A\partial_\zeta(Rf)\right)}{ARf} + \frac{\partial_\zeta\left(A\partial_\zeta(Rg)\right)}{ARg}\right]$$

Notice that in this form h''/h is a single-variable function. Since any non-constant multiplier would complicate either f''/f or g''/g, each of the summands must also be functions of a single variable or else $\Delta^*\psi$ will not separate.

Explicitly, the leading terms must be sums of some single variable functions $f^*(\zeta)$ and $g^*(\eta)$,

$$\frac{\partial_\zeta A \partial_\zeta R}{AR} + \frac{\partial_\zeta^2 R}{R} + \frac{\partial_\eta A \partial_\eta R}{AR} + \frac{\partial_\eta^2 R}{R} = f^*(\zeta) + g^*(\eta) \tag{3.10}$$

By the same logic, the h'/h terms, which are more restricted, must also reduce to single-variable functions. If it is desired to find an appropriate R without prior knowledge, these terms may be more approachable to start.

That is, $R(\zeta, \eta)$ must be chosen so that for some f^{**} and g^{**},

$$\frac{\partial_\zeta A}{A} + 2\frac{\partial_\zeta R}{R} = f^{**}(\zeta), \qquad \frac{\partial_\eta A}{A} + 2\frac{\partial_\eta R}{R} = g^{**}(\eta)$$

One solution for R is given in [6] and [7] as

$$R(\zeta, \eta) = \frac{\sinh \zeta}{\sqrt{\cosh \zeta - \cos \eta}} \tag{3.11}$$

which can be verified as appropriate by replacement. This further refines equation 3.8,

$$\begin{aligned} \Delta^*\psi &= \frac{(\cosh \zeta - \cos \eta)^2}{r_0^3} ARfg \left[\frac{\partial_\zeta \left(A \partial_\zeta (Rf) \right)}{ARf} + \frac{\partial_\zeta \left(A \partial_\zeta (Rg) \right)}{ARg} \right] \\ &= \frac{(\cosh \zeta - \cos \eta)^{5/2}}{r_0^3} fg \left[\frac{\partial_\zeta \left(A \partial_\zeta (Rf) \right)}{ARf} + \frac{\partial_\zeta \left(A \partial_\zeta (Rg) \right)}{ARg} \right] \end{aligned}$$

The replacement is then verified in full, where tedious calculations are relegated to appendix B.

First compute $f^{**}(\zeta)$ where the components are

$$\frac{\partial_\zeta A}{A} = \frac{\cosh\zeta\cos\eta - 1}{(\cosh\zeta - \cos\eta)\sinh\zeta}, \qquad \frac{\partial_\zeta R}{R} = \frac{\cosh^2\zeta - 2\cosh\zeta\cos\eta + 1}{2(\cosh\zeta - \cos\eta)\sinh\zeta}$$

Then

$$f^{**}(\zeta) = \frac{\partial_\zeta A}{A} + 2\frac{\partial_\zeta R}{R} = \frac{\cosh^2\zeta - \cosh\zeta\cos\eta}{(\cosh\zeta - \cos\eta)\sinh\zeta} = \frac{\cosh\zeta}{\sinh\zeta}$$

which is indeed a function of a single variable.

Similarly for $g^{**}(\eta)$,

$$\begin{aligned} \partial_\eta A(\zeta,\eta) &= \frac{\sin\eta}{\sinh\zeta}, & \partial_\eta R(\zeta,\eta) &= \frac{-\sinh\zeta\sin\eta}{2(\cosh\zeta - \cos\eta)^{3/2}} \\ \frac{\partial_\eta A}{A} &= \frac{\sin\eta}{\cosh\zeta - \cos\eta}, & \frac{\partial_\eta R}{R} &= \frac{-\sin\eta}{2(\cosh\zeta - \cos\eta)} \end{aligned}$$

so that

$$g^{**}(\eta) = \frac{\partial_\eta A}{A} + 2\frac{\partial_\eta R}{R} = 0$$

Using these results condenses 3.8 even further,

$$\Delta^*\psi = \frac{(\cosh\zeta - \cos\eta)^{5/2}}{r_0^3} fg\left[f^* + \frac{f'\cosh\zeta}{f\sinh\zeta} + \frac{f''}{f} + g^* + \frac{g''}{g}\right]$$

Then, for the other unknown terms, revisit equation 3.10,

$$\frac{\partial_\zeta A\partial_\zeta R}{AR} + \frac{\partial_\zeta^2 R}{R} + \frac{\partial_\eta A\partial_\eta R}{AR} + \frac{\partial_\eta^2 R}{R} = f^*(\zeta) + g^*(\eta) \qquad \text{(3.10 Revisited)}$$

and proceed to compute each of the component summands.

Using the results from above,

$$\begin{aligned}
\frac{\partial_\zeta A \partial_\zeta R}{AR} &= \left(\frac{\cosh\zeta\cos\eta - 1}{(\cosh\zeta - \cos\eta)\sinh\zeta}\right)\left(\frac{\cosh^2\zeta - 2\cosh\zeta\cos\eta + 1}{2(\cosh\zeta - \cos\eta)\sinh\zeta}\right) \\
&= \frac{1}{2} + \frac{\sin^2\eta}{2(\cosh\zeta - \cos\eta)^2} - \frac{\cosh^2\zeta}{\sinh^2\zeta} + \frac{\cosh\zeta}{2(\cosh\zeta - \cos\eta)} \\
\frac{\partial_\eta A \partial_\eta R}{AR} &= \frac{-\sin^2\eta}{2(\cosh\zeta - \cos\eta)^2}
\end{aligned}$$

so that the combined partial sum becomes,

$$\frac{\partial_\zeta A \partial_\zeta R}{AR} + \frac{\partial_\zeta A \partial_\zeta R}{AR} = \frac{1}{2} - \frac{\cosh^2\zeta}{\sinh^2\zeta} + \frac{\cosh\zeta}{2(\cosh\zeta - \cos\eta)}$$

For the remaining terms of equation 3.10, one more of each partial derivative must be taken for R,

$$\begin{aligned}
\partial_\zeta^2 R &= \frac{\sinh\zeta}{4(\cosh\zeta - \cos\eta)^{5/2}}\left((\cosh\zeta - \cos\eta)^2 - 3\sin^2\eta\right) \\
\partial_\eta^2 R &= \frac{\sinh\zeta}{4(\cosh\zeta - \cos\eta)^{5/2}}\left(3\sin^2\eta - 2(\cosh\zeta - \cos\eta)\cos\eta\right)
\end{aligned}$$

so that together

$$\frac{\partial_\zeta^2 R + \partial_\eta^2 R}{R} = \frac{1}{4} - \frac{\cos\eta}{2(\cosh\zeta - \cos\eta)}$$

and combining the results,

$$f^*(\zeta) + g^*(\eta) = \frac{5}{4} - \frac{\cosh^2\zeta}{\sinh^2\zeta} = \frac{1}{4} - \frac{1}{\sinh^2\zeta}$$

Then with the separation, equation 3.8 can be written

$$\Delta^*\psi = \frac{(\cosh\zeta - \cos\eta)^{5/2}}{r_0^3} fg\left[\frac{1}{4} - \frac{1}{\sinh^2\zeta} + \frac{f'\cosh\zeta}{f\sinh\zeta} + \frac{f''}{f} + \frac{g''}{g}\right]$$

so in homogeneous form,

$$0 = \frac{(\cosh\zeta - \cos\eta)^{5/2}}{r_0^3} fg \left[\frac{1}{4} - \frac{1}{\sinh^2\zeta} + \frac{f'\cosh\zeta}{f\sinh\zeta} + \frac{f''}{f} + \frac{g''}{g}\right]$$

Since ($\cosh\zeta = \cos\eta$) only at the foci, $(\pm r_0, z_0)$, either fg is identically zero, implying ψ is identically zero as well, or for some constant c both of the following hold,

$$\frac{f''}{f} + \frac{f'\cosh\zeta}{f\sinh\zeta} - \frac{1}{\sinh^2\zeta} + \frac{1}{4} = c, \qquad \frac{g''}{g} = -c \tag{3.12}$$

This is the point at which 3.8 is considered fully “separated,” and since for the scenario in question ψ is not identically zero, 3.12 must hold.

Subsection 3.2.1: Separated Solutions

For $g'' = -cg$, the general solution is familiar enough to state directly, where for some constants α and β,

$$g(\eta) = \alpha\cos(\eta\sqrt{c}) + \beta\sin(\eta\sqrt{c}) \tag{3.13}$$

Interested readers that would like to solve this equation without hand-waving can consult differential equations texts such as [14] or [15].

The idea is to reduce the second order equation to a two-dimensional first order equation by introducing a new variable $w = g'$, so that $w' = g''$, and in matrix form,

$$\begin{bmatrix} g' \\ w' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ c & 0 \end{bmatrix} \begin{bmatrix} g \\ w \end{bmatrix}$$

This is then solved using eigenvector decomposition.

The general solution listed in 3.13 is not the final representation however. Notice that η is a periodic coordinate, so that for any $n \in \mathbb{Z}$, it must be that $g(\eta) = g(\eta + 2\pi n)$.

Thus,

$$\begin{aligned} g(\eta + 2\pi n) &= \alpha \cos\left((\eta + 2\pi n)\sqrt{c}\right) + \beta \sin\left((\eta + 2\pi k)\sqrt{c}\right) \\ &= \alpha \cos\left(\eta\sqrt{c} + 2\pi n\sqrt{c}\right) + \beta \sin\left(\eta\sqrt{c} + 2\pi n\sqrt{c}\right) \\ &= \alpha \cos\left(\eta\sqrt{c}\right) + \beta \sin\left(\eta\sqrt{c}\right) = g(\eta) \end{aligned}$$

If $\sqrt{c}$ is anything but a non-negative integer (consider only positive value of the square root for this solution), then the identity will only hold if g is identically zero, making ψ vanish everywhere. Thus $\sqrt{c}$ must be a non-negative integer to obtain any remaining solutions, $\sqrt{c} = k \in \mathbb{Z}_+$. These can be summed together for the full result,

$$g(\eta) = \sum_{k=0}^{\infty} \alpha_k \cos(k\eta) + \beta_k \sin(k\eta) \tag{3.14}$$

A little more work has to be put into the other equation. So, replacing $c = k^2$,

$$\begin{aligned} f'' + \left(\frac{\cosh\zeta}{\sinh\zeta}\right) f' + \left(\frac{1}{4} - k^2 - \frac{1}{\sinh^2\zeta}\right) f = 0 \\ -f'' - \left(\frac{\cosh\zeta}{\sinh\zeta}\right) f' - \left(\frac{1}{4} - k^2 - \frac{1}{\sinh^2\zeta}\right) f = 0 \\ \left(\frac{1}{\sinh\zeta}\right) \frac{d}{d\zeta}\left((-\sinh\zeta)\frac{df}{d\zeta}\right) + \left(k^2 - \frac{1}{4} - \frac{1}{-\sinh^2\zeta}\right) f = 0 \end{aligned}$$

Rather than solve this equation directly, use a change of variables, $u = \cosh\zeta$ so that

$$\frac{d}{d\zeta} = (\sinh\zeta)\frac{d}{du}$$

and

$$\begin{aligned}
\left(\frac{1}{\sinh\zeta}\right)\frac{d}{d\zeta}\left((-\sinh\zeta)\frac{df}{d\zeta}\right)+\left(k^2-\frac{1}{4}-\frac{1}{-\sinh^2\zeta}\right)f=0\\
\frac{d}{du}\left((-\sinh^2\zeta)\frac{df}{du}\right)+\left(k^2-\frac{1}{4}-\frac{1}{-\sinh^2\zeta}\right)f=0\\
\frac{d}{du}\left((1-\cosh^2\zeta)\frac{df}{du}\right)+\left(k^2-\frac{1}{4}-\frac{1}{1-\cosh^2\zeta}\right)f=0\\
\frac{d}{du}\left((1-u^2)\frac{df}{du}\right)+\left(\left(k-\frac{1}{2}\right)\left(k+\frac{1}{2}\right)-\frac{1}{1-u^2}\right)f=0
\end{aligned}$$

This approach is useful because it reveals a specific case of the associated Legendre equation,

$$\frac{d}{du}\left((1-u^2)\frac{df}{du}\right)+\left(l(l+1)-\frac{m}{1-u^2}\right)f=0$$

where $l=\left(k-\frac{1}{2}\right)$ and $m=1$.

The associated Legendre equation has a known general solution from [13], [16], [17], and others. It is written here assuming the notation from [18] with leading constants a and b,

$$\begin{aligned}
f(\zeta)&=aP_l^m(u)+bQ_l^m(u)\\
f(\zeta)&=a_kP^1_{k-\frac{1}{2}}(\cosh\zeta)+b_kQ^1_{k-\frac{1}{2}}(\cosh\zeta)
\end{aligned}$$

Then, together with 3.14, the full separated form must account for each potential value of k and is written

$$\psi(\zeta,\eta)=\frac{\sinh\zeta}{\sqrt{\cosh\zeta-\cos\eta}}\sum_{k=0}^{\infty}\left(a_kP^1_{k-\frac{1}{2}}(\cosh\zeta)+b_kQ^1_{k-\frac{1}{2}}(\cosh\zeta)\right)(\alpha_k\cos(k\eta)+\beta_k\sin(k\eta))$$

where distributing the product gives,

$$\psi(\zeta,\eta) = \frac{\sinh\zeta}{\sqrt{\cosh\zeta - \cos\eta}} \sum_{k=0}^{\infty} \Big(a_k\alpha_k P^1_{k-\frac{1}{2}}(\cosh\zeta)\cos(k\eta) + b_k\alpha_k Q^1_{k-\frac{1}{2}}(\cosh\zeta)\cos(k\eta) \Big)$$
$$+ \Big(a_k\beta_k P^1_{k-\frac{1}{2}}(\cosh\zeta)\sin(k\eta) + b_k\beta_k Q^1_{k-\frac{1}{2}}(\cosh\zeta)\sin(k\eta) \Big)$$

Combining the constants so that

$$A_k = a_k\alpha_k, \qquad B_k = b_k\alpha_k, \qquad C_k = a_k\beta_k, \qquad D_k = b_k\beta_k$$

gives the most usable form,

$$\psi(\zeta,\eta) = \frac{\sinh\zeta}{\sqrt{\cosh\zeta - \cos\eta}} \sum_{k=0}^{\infty} \Big(A_k P^1_{k-\frac{1}{2}}(\cosh\zeta)\cos(k\eta) + B_k Q^1_{k-\frac{1}{2}}(\cosh\zeta)\cos(k\eta) \Big)$$
$$+ \Big(C_k P^1_{k-\frac{1}{2}}(\cosh\zeta)\sin(k\eta) + D_k Q^1_{k-\frac{1}{2}}(\cosh\zeta)\sin(k\eta) \Big) \tag{3.15}$$

CHAPTER 4: APPLICATION

After a general form for ψ has been found in equation 3.15, a specific solution is approximated by optimizing it to fit available measurements from the tokamak itself. There are a large variety of measuring devices available, and each is related to ψ in a different way.

Common probes types are shown, while the most directly useful is applied. The parameters needed to effectively apply measurements are incorporated so that the method by which they are used can be described. A simple example is given for context and then generalized.

Section 4.1: Probes

Probes on a tokamak come in a variety of types, as described in [19]. Each measures some characteristic of the electromagnetic field surrounding the plasma, and with the right formatting can be used for optimizing the specific solution of 3.15.

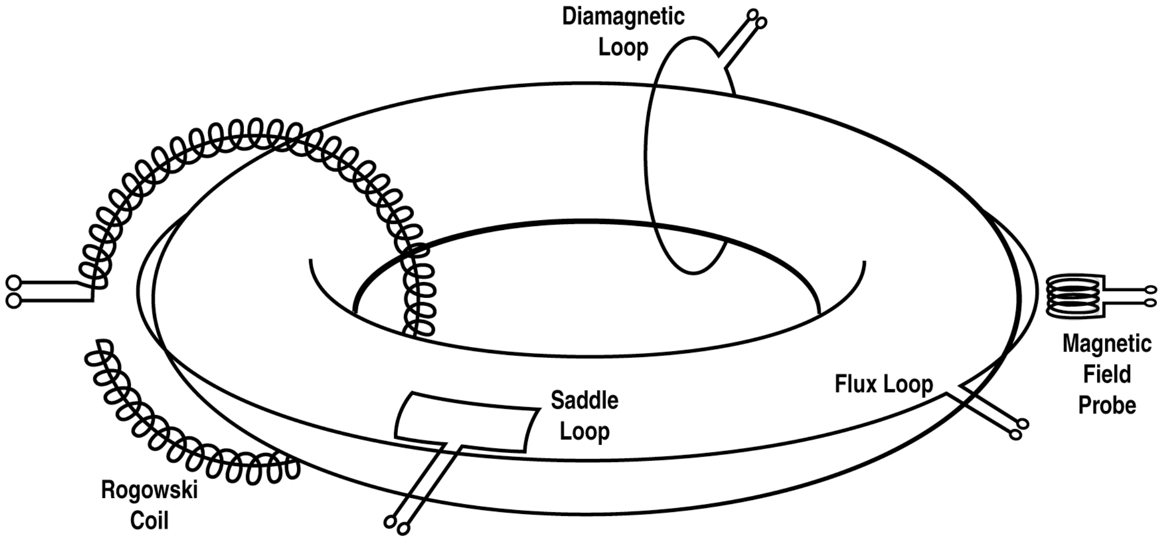

Figure 4.1: Tokamak probes from [19]

The most direct measurements for this task are from flux loops. As pictured in figure 4.1, these run the toroidal circumference of the machine, and thus appear as points in the poloidal plane. Flux loops measure the poloidal magnetic flux, ψ, at their (r, z) location directly, and thus can be used without additional refinement.

Due to the nature of their mechanics, there are often only one or two flux loops on a given machine. Other measurements that are taken as from a flux loop are actually difference measurements given by saddle loops. These allow a measurement of the difference in flux between two poloidal (r, z) points. When used in conjunction with a direct measurement these can act as flux loops themselves in that the second poloidal point for which they "saddle" has a poloidal flux equal to the reference measurement plus the difference measurement from the saddle loop.

This subtlety is mostly glossed over here and in a machine's hardware, and flux data at any poloidal point is given in absolute terms as if it came from a direct measurement. Altering the particular definitions from a specific tokamak only requires knowing which are the reference probes if the saddle measurements haven't been adjusted as such already.

Including even more types of data, such as magnetic field probe measurements, improves the accuracy of the approximation by allowing for more redundancy. The other values that result are still related to the poloidal flux, and thus they tend to be converted to accommodate needed form a priori. Since this takes a little more explanation for each probe type, and the flux probes are the main measurements applied in the optimization here, additional probe data is left to be incorporated in future work.

The type of probe itself is less important than its location in relation to the vacuum. The final result from equation 3.15 applies only to the probes when there is nothing but vacuum between the probes and the plasma, and if probes are located outside the vacuum vessel of the machine, the data from them requires significant alteration to be usable.

This adjustment is needed because the vessel itself can carry a non-negligible current that will shield the outside probes and prevent accurate data. Thus probes external to the machine are rendered mostly useless without additional considerations that will not be covered here.

Subsection 4.1.1: Data

Algorithms used to find the boundary of the plasma are normally applied in real time, but during their development prior experimental data must be used to verify their accuracy. Most tokamaks record all their experiments in large databases which prove useful for this purpose.

Much of this data may be proprietary, or at least restricted to a select group of people, and care must be taken to acquire the necessary permissions to use it. Publicly managed devices and tokamaks in the reader's home country may be more willing to share such records, and it is recommended to inquire with them initially.

Specific tokamak aside, the data should all be roughly of the same type after accounting for formatting and units. Once the data is accessible, measured values of ψ at a number of collection points in the poloidal plane can be used. With flux probes these measurements are direct, and only the poloidal locations of the probes are needed in addition to the measurements themselves. This makes them good first candidates for testing new algorithms.

Consider some set of N total flux probes with known locations $(r_i, z_i) = (\zeta_i, \eta_i)_{i=1}^N$ each with a measurement ψ_i^* that has been recorded and formatted so that a list of values can be compiled,

$$(\psi_i^*, \zeta_i, \eta_i)_{i=1}^N = \{(\psi_1^*, \zeta_1, \eta_1), \ldots, (\psi_N^*, \zeta_N, \eta_N)\} \tag{4.1}$$

It is assumed these measurements are taken simultaneously so that they must all fit the form from 3.15 with the same coefficients. This allows for the coefficients to be optimized to fit the measurements as best as possible using least squares. With the resulting values, ψ can be characterized anywhere in the vacuum, and specifically at its edge. Then since the plasma is confined to constant flux surfaces, the outermost closed contours of the optimized form must be these edges, and they are taken as the boundaries of the plasma's containment.

Section 4.2: Method

Suppose there are N simultaneous measurements from flux loops on a tokamak at known locations. Collected in ordered triplets they can be listed $(\psi_i^*, \zeta_i, \eta_i)_{i=1}^N$. Moreover it is assumed that conditions from this thesis hold so that the representation of the flux matches that of equation 3.15, repeated here:

$$\psi(\zeta,\eta) = \frac{\sinh\zeta}{\sqrt{\cosh\zeta - \cos\eta}} \sum_{k=0}^{\infty} \Big(A_k P^1_{k-\frac{1}{2}}(\cosh\zeta)\cos(k\eta) + B_k Q^1_{k-\frac{1}{2}}(\cosh\zeta)\cos(k\eta)\Big) + \Big(C_k P^1_{k-\frac{1}{2}}(\cosh\zeta)\sin(k\eta) + D_k Q^1_{k-\frac{1}{2}}(\cosh\zeta)\sin(k\eta)\Big)$$

Each component of the sum contributes four unknown coefficients, so that if the sum were truncated to some maximum number of terms, $M \in \mathbb{Z}_+$,

$$\tilde{\psi}(\zeta,\eta) = \frac{\sinh\zeta}{\sqrt{\cosh\zeta - \cos\eta}} \sum_{k=0}^{M} \Big(A_k P^1_{k-\frac{1}{2}}(\cosh\zeta)\cos(k\eta) + B_k Q^1_{k-\frac{1}{2}}(\cosh\zeta)\cos(k\eta)\Big) + \Big(C_k P^1_{k-\frac{1}{2}}(\cosh\zeta)\sin(k\eta) + D_k Q^1_{k-\frac{1}{2}}(\cosh\zeta)\sin(k\eta)\Big) \tag{4.2}$$

since $\sin(0) = 0$, two coefficients can be removed as free variables, and in total there must be $2(2M-1)$ measurements at distinct $\cosh\zeta$ and η coordinates to provide a unique solution.

It is assumed that $\psi_i^* \approx \tilde{\psi}(\zeta_i, \eta_i)$ for each i so that the truncated representation 4.2 can be used to limit the number of unknowns. While this may be an approximation at best, the over-constrained problem that results from $N > 2(2M-1)$ should tend to remove any small inaccuracies.

Any over-constrained problem of this nature allows for some error, as a least squares optimization procedure,

$$\min \sum_{i=1}^{N} (\tilde{\psi} - \psi_i^*)^2 \tag{4.3}$$

finds the most appropriate coefficients to match all measurements as close as possible regardless of form.

Once the coefficients have been solved for, a line search procedure in conjunction with a method to check for closure can check values of ψ to find the largest closed flux surfaces. These are the desired boundaries of the plasma.

Subsection 4.2.1: Linear Decomposition

The form from equation 3.15 is useful because it is linear in terms of the coefficients. Because the measurements do not change position, the toroidal harmonics can be evaluated at the location of each once and then stored in a table. Then the problem is simple linear programming, as described here.

Unique Solution Example

Consider a simple example where there are exactly 2 measurements about the pole ($r_0 = 1, z_0 = 0$), both located directly on the same radial axis,

$$
\begin{aligned}
&(\psi_1^* = 1, r_1 = 3/2, z_1 = 0), && (\psi_2^* = 1, r_2 = 1/2, z_2 = 0)\\
&(\zeta_1 \approx 1.60944, \eta_1 = 0), && (\zeta_2 \approx 1.0986, \eta_2 = \pi)\\
&\cosh\zeta_1 \approx 2.6, && \cosh\zeta_2 \approx 1.6667\\
&\sinh\zeta_1 \approx 2.4, && \sinh\zeta_2 \approx 1.3333\\
&\cos\eta_1 = 1, && \cos\eta_2 = -1
\end{aligned}
$$

Then for one term of the sum, ie. $M = 1$ in equation 4.2, the coefficients can be computed uniquely. This is because $\cos(0\eta) = 1$, $\sin(0\eta) = 0$,

$$
\begin{aligned}
&P^1_{-1/2}(\cosh\zeta_1) \approx 0.15359, && P^1_{-1/2}(\cosh\zeta_2) \approx 0.12035\\
&Q^1_{-1/2}(\cosh\zeta_1) \approx 0.73909, && Q^1_{-1/2}(\cosh\zeta_2) \approx 1.04881
\end{aligned}
$$

and

$$
\frac{\sinh\zeta_1}{\sqrt{\cosh\zeta_1 - \cos\eta_1}} \approx 1.89737, \qquad \frac{\sinh\zeta_2}{\sqrt{\cosh\zeta_2 - \cos\eta_2}} \approx 0.81650
$$

so that only two coefficients remain and the k subscripts can be dropped. In this scenario

$$
\tilde{\psi}(\zeta, \eta) = \frac{\sinh\zeta}{\sqrt{\cosh\zeta - \cos\eta}} \left(AP^1_{-\frac{1}{2}}(\cosh\zeta) + BQ^1_{-\frac{1}{2}}(\cosh\zeta) \right)
$$

By assuming $\psi_i^* = \tilde{\psi}(\zeta_i, \eta_i)$ as before and using the computations from above, a linear problem results,

$$\begin{bmatrix} \psi_1^* \\ \psi_2^* \end{bmatrix} = \operatorname{diag}\left(\frac{\sinh \zeta_1}{\sqrt{\cosh \zeta_1 - \cos \eta_1}}, \frac{\sinh \zeta_2}{\sqrt{\cosh \zeta_2 - \cos \eta_2}} \right) \begin{bmatrix} P^1_{-\frac{1}{2}}(\cosh \zeta_1) & Q^1_{-\frac{1}{2}}(\cosh \zeta_1) \\ P^1_{-\frac{1}{2}}(\cosh \zeta_2) & Q^1_{-\frac{1}{2}}(\cosh \zeta_2) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \operatorname{diag}(1.89737, 0.81650) \begin{bmatrix} 0.15359 & 0.73909 \\ 0.12035 & 1.04881 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \operatorname{diag}\left(1.89737^{-1}, 0.81650^{-1}\right) \begin{bmatrix} 0.15359 & 0.73909 \\ 0.12035 & 1.04881 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{\operatorname{diag}(0.52705, 1.22474)}{0.07214} \begin{bmatrix} 1.04881 & -0.73909 \\ -0.12035 & 0.15359 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 2.26278 \\ 0.56433 \end{bmatrix}$$

giving

$$\tilde{\psi}(\zeta, \eta) = \frac{\sinh \zeta}{\sqrt{\cosh \zeta - \cos \eta}} \left((2.26278) P^1_{-\frac{1}{2}}(\cosh \zeta) + (0.56433) Q^1_{-\frac{1}{2}}(\cosh \zeta) \right)$$

This A and B represent the unique solution to match the measurements exactly with the given form. This is the least constrained the problem can be and still give a unique solution, but because of the lack of over-constraint, the representation is prone to error given variations in the measurements.

For instance, if the external measurement was instead 1.1, the resulting coefficients would be

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{\operatorname{diag}(0.52705, 1.22474)}{0.07214} \begin{bmatrix} 1.04881 & -0.73909 \\ -0.12035 & 0.15359 \end{bmatrix} \begin{bmatrix} 1.1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 3.02905 \\ 0.36000 \end{bmatrix}$$

so that a 10% difference in one measurement produces more than a 30% change in the coefficients.

This is obviously anecdotal evidence, but more measurements are nevertheless preferable; they "smooth out" small errors in individual measurements and give a more robust solution.

Generalization

To expand on the technique from the previous example, consider the results for some arbitrary M where $N > 2(M-1)$ so that the system is over-constrained. Even without the truncated form, given ψ from equation 3.15,

$$\psi(\zeta,\eta) = \frac{\sinh\zeta}{\sqrt{\cosh\zeta - \cos\eta}} \sum_{k=0}^{\infty} \Big(A_k P^1_{k-\frac{1}{2}}(\cosh\zeta)\cos(k\eta) + B_k Q^1_{k-\frac{1}{2}}(\cosh\zeta)\cos(k\eta) \Big)$$
$$+ \Big(C_k P^1_{k-\frac{1}{2}}(\cosh\zeta)\sin(k\eta) + D_k Q^1_{k-\frac{1}{2}}(\cosh\zeta)\sin(k\eta) \Big)$$

the same procedure can be understood more thoroughly by re-writing the equation in matrix form.

So let $w(\zeta,\eta)$ represent the leading hyperbolic coefficient, $T_k(\zeta,\eta)$ the toroidal harmonic vector, and $\mathbf{T}(\zeta,\eta)$ the combined harmonic vector,

$$w = \frac{\sinh\zeta}{\sqrt{\cosh\zeta - \cos\eta}}, \qquad T_k = \begin{bmatrix} P^1_{k-\frac{1}{2}}(\cosh\zeta)\cos(k\eta) \\ Q^1_{k-\frac{1}{2}}(\cosh\zeta)\cos(k\eta) \\ P^1_{k-\frac{1}{2}}(\cosh\zeta)\sin(k\eta) \\ Q^1_{k-\frac{1}{2}}(\cosh\zeta)\sin(k\eta) \end{bmatrix}, \qquad \mathbf{T} = \begin{bmatrix} T_0 \\ T_1 \\ \vdots \end{bmatrix}$$

Also take U to be the unknown coefficient vector, and $\mathbf{U}$ to be the combined unknown vector,

$$U_k = \begin{bmatrix} A_k \\ B_k \\ C_k \\ D_k \end{bmatrix} \qquad \mathbf{U} = \begin{bmatrix} U_0 \\ U_1 \\ \vdots \end{bmatrix}$$

The whole equation from 3.15 then condenses down to

$$\psi(\zeta,\eta) = w(\zeta,\eta)\mathbf{T}^T(\zeta,\eta)\mathbf{U} \tag{4.4}$$

illustrating the linear nature of the problem.

By assuming $\psi_i^* \approx \psi$, for each measurement i, this gives a system of equations that can be written in a similar convention. The unknowns do not change since the coefficients from 3.15 must be the same regardless of location, and then the variables can be subscripted

$$w_i = \frac{\sinh\zeta_i}{\sqrt{\cosh\zeta_i - \cos\eta_i}} \qquad (T_k)_i = \begin{bmatrix} P^1_{k-\frac{1}{2}}(\cosh\zeta_i)\cos(k\eta_i) \\ Q^1_{k-\frac{1}{2}}(\cosh\zeta_i)\cos(k\eta_i) \\ P^1_{k-\frac{1}{2}}(\cosh\zeta_i)\sin(k\eta_i) \\ Q^1_{k-\frac{1}{2}}(\cosh\zeta_i)\sin(k\eta_i) \end{bmatrix} \qquad \mathbf{T}_i = \begin{bmatrix} (T_0)_i \\ (T_1)_i \\ \vdots \end{bmatrix}$$

Namely, for N measurements as proposed, and $\psi_i^* \approx \psi$,

$$\begin{bmatrix} \psi_1^* \\ \vdots \\ \psi_N^* \end{bmatrix} = \operatorname{diag}(w_1, \ldots, w_N) \begin{bmatrix} \mathbf{T}_1^T \\ \vdots \\ \mathbf{T}_N^T \end{bmatrix} \mathbf{U} \tag{4.5}$$

With sufficiently short $\mathbf{U}$, ie. for a truncated set of harmonics where $\psi \approx \tilde{\psi}$ and $N > 2(2M-1)$, call

$$\tilde{\mathbf{T}} = \begin{bmatrix} T_0 \\ \vdots \\ T_M \end{bmatrix}, \qquad \tilde{\mathbf{U}} = \begin{bmatrix} U_0 \\ \vdots \\ U_M \end{bmatrix}$$

so the least squares regression proposed in equation 4.3 can be applied to find the optimal $\mathbf{U}$,

$$\min \sum_{i=1}^{N} \left(\tilde{\psi}(\zeta_i, \eta_i) - \psi^*(\zeta_i, \eta_i) \right)^2 \tag{4.6}$$
$$\tilde{\psi} = w \tilde{\mathbf{T}}^T \tilde{\mathbf{U}}$$

This is the crux of the decomposition of ψ, and this method as a whole. It is an unconstrained linear programming problem, and thus can be solved using the simplex method, Karmarkar's algorithm [20], or the reader's linear optimization algorithm of choice. Interested readers are referred to [21] for an introduction into optimization algorithms.

Once the coefficients have been optimized, the final step is determining the largest closed contours of the resulting representation of ψ. Since the current in the plasma is restricted to constant flux surfaces, closed contours form boundaries that contain the plasma, while open contours allow it to dissipate into contacting surfaces. For this reason, the largest closed contours are the last surfaces to contain the plasma, and only vacuum surrounds them.

These boundaries are thus limiting points of the vacuum region, and the form of ψ is still valid right up to their edge. They can be found using a standard line search algorithm (again in [21] if needed) in conjunction with any function that checks for closure of the resulting level curves. The appropriate constant values of ψ then correspond with the bounding contours as desired.

CHAPTER 5: CONCLUSIONS

Once the boundary of ψ has been found, so has the corresponding constant value of ψ on it. By applying these boundary conditions, the Grad-Shafranov equation,

$$-\Delta^*\psi = \mu r P'(\psi) + \frac{T(\psi)}{r} T'(\psi) \tag{2.24 Revisited}$$

can be solved with standard fixed-boundary techniques, such as finite elements as used in [22], and described as a technique in [23].

The plasma pressure and magnetic characteristics fall out of this equation since $T = rB_\phi$ and

$$\mathbf{B} = r^{-1}\left(T\hat{\boldsymbol{\phi}} + \nabla\psi \times \hat{\boldsymbol{\phi}}\right) \tag{2.18 Revisited}$$

Moreover, the internal current profile can then be computed once T has been found by way of the complete inhomogeneous equation,

$$\mathbf{J} = -\mu^{-1}\Delta^*\psi\hat{\boldsymbol{\phi}} + \frac{\nabla T}{\mu r} \times \hat{\boldsymbol{\phi}}$$

which can again be solved using fixed boundary techniques.

It is important to note that when making use of the method from this paper, it is important that all of the required assumptions are met. This includes axis-symmetry, perfect conductivity, and the time-scale assumptions, but more importantly, the locations of the measurements must be withing the vacuum.

Any conductive material that cuts off the vacuum surrounding the plasma from the measuring devices will carry a non-negligible current that invalidates the majority of the results. Care must also be taken with the half-order toroidal harmonics P and Q as they have a variety of both real and complex variations in the literature. Here they are taken as in [18] since it is the most commonly available code that computes them.

While there is code that can accomplish many of the tasks presented, because of complications that arose in their implementation, their use must be proposed in further work. Much of the entire process given here has been validated in other sources ([7],[6]), but the actual code needed to do so has yet to be made freely available.

Further work should focus on publishing such algorithms that compile both the formatting process of different measuring devices, as well as accounting for the toroidal harmonics needed to produce a sufficiently robust solution.

Explanations of their use should be written such that they are within the realm of the technically proficient without being exclusive to the computer science or professional programming community. Presented in a simple language like MATLAB, or even as pseudocode, it would be able to reach a wide audience that may open the door to further understanding and progress in the realm of boundary reconstruction algorithms.

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APPENDIX A: COORDINATE DERIVATIONS

Restating the toroidal coordinate definitions from section 3.1 gives a good starting point for the derivations. So again consider a point using cylindrical coordinates (r, z) in the right half plane, along with some given foci $(\pm r_0, z_0)$.

The corresponding toroidal coordinates (ζ, η) are given by first computing the corresponding distances to the foci,

$$d_- = \sqrt{(r + r_0)^2 + (z - z_0)^2} \qquad\qquad d_+ = \sqrt{(r - r_0)^2 + (z - z_0)^2}$$

so that the corresponding coordinates are then,

$$\zeta = \ln \frac{d_-}{d_+} \qquad\qquad \eta = \arccos\left(\frac{d_-^2 + d_+^2 - 4r_0^2}{2d_- d_+}\right) \tag{3.2}$$

with inverse transformations,

$$r = \frac{r_0 \sinh \zeta}{\cosh \zeta - \cos \eta} \qquad\qquad z = z_0 + \frac{r_0 \sin \eta}{\cosh \zeta - \cos \eta} \tag{3.3}$$

The transformations from 3.3 are then used to put d_- and d_+ in hyperbolic terms,

$$\begin{aligned}
d_- &= \sqrt{\left(\frac{r_0 \sinh\zeta}{\cosh\zeta - \cos\eta} + r_0\right)^2 + \left(\frac{r_0 \sin\eta}{\cosh\zeta - \cos\eta}\right)^2} \\
&= r_0 \sqrt{\frac{(\sinh\zeta + \cosh\zeta - \cos\eta)^2 + \sin^2\eta}{(\cosh\zeta - \cos\eta)^2}} \\
&= r_0 \sqrt{\frac{\sinh^2\zeta + (\cosh\zeta - \cos\eta)^2 + 2\sinh\zeta(\cosh\zeta - \cos\eta) + \sin^2\eta}{(\cosh\zeta - \cos\eta)^2}} \\
&= r_0 \sqrt{\frac{\sinh^2\zeta + \cosh^2\zeta + \cos^2\eta - 2\cosh\zeta\cos\eta + 2\sinh\zeta(\cosh\zeta - \cos\eta) + \sin^2\eta}{(\cosh\zeta - \cos\eta)^2}} \\
&= r_0 \sqrt{\frac{\sinh^2\zeta + \cosh^2\zeta + 1 - 2\cosh\zeta\cos\eta + 2\sinh\zeta(\cosh\zeta - \cos\eta)}{(\cosh\zeta - \cos\eta)^2}} \\
&= r_0 \sqrt{\frac{2\cosh^2\zeta - 2\cosh\zeta\cos\eta + 2\sinh\zeta(\cosh\zeta - \cos\eta)}{(\cosh\zeta - \cos\eta)^2}} \\
&= r_0 \sqrt{\frac{2(\cosh\zeta + \sinh\zeta)(\cosh\zeta - \cos\eta)}{(\cosh\zeta - \cos\eta)^2}} = r_0 \sqrt{2\frac{\cosh\zeta + \sinh\zeta}{\cosh\zeta - \cos\eta}}
\end{aligned}$$

$$\begin{aligned}
d_+ &= \sqrt{\left(\frac{r_0 \sinh\zeta}{\cosh\zeta - \cos\eta} - r_0\right)^2 + \left(\frac{r_0 \sin\eta}{\cosh\zeta - \cos\eta}\right)^2} \\
&= r_0 \sqrt{\frac{(\sinh\zeta - \cosh\zeta + \cos\eta)^2 + \sin^2\eta}{(\cosh\zeta - \cos\eta)^2}} \\
&= r_0 \sqrt{\frac{(\sinh\zeta - \cosh\zeta)^2 + 2(\sinh\zeta - \cosh\zeta)\cos\eta + 1}{(\cosh\zeta - \cos\eta)^2}} \\
&= r_0 \sqrt{\frac{\sinh^2\zeta - 2\cosh\zeta\sinh\zeta + \cosh^2\zeta + 2(\sinh\zeta - \cosh\zeta)\cos\eta + 1}{(\cosh\zeta - \cos\eta)^2}} \\
&= r_0 \sqrt{\frac{2\cosh^2\zeta - 2\cosh\zeta\sinh\zeta + 2(\sinh\zeta - \cosh\zeta)\cos\eta}{(\cosh\zeta - \cos\eta)^2}} \\
&= r_0 \sqrt{\frac{2(\cosh\zeta - \sinh\zeta)(\cosh\zeta - \cos\eta)}{(\cosh\zeta - \cos\eta)^2}} = r_0 \sqrt{2\frac{\cosh\zeta - \sinh\zeta}{\cosh\zeta - \cos\eta}}
\end{aligned}$$

For the partial derivatives in hyperbolic terms, it is convenient to notice that

$$\begin{aligned}
r - r_0 &= \frac{r_0 \sinh\zeta}{\cosh\zeta - \cos\eta} - r_0 = \frac{r_0(\sinh\zeta - \cosh\zeta + \cos\eta)}{\cosh\zeta - \cos\eta} \\
&= r_0\left(\frac{\cos\eta}{\cosh\zeta - \cos\eta} - \frac{\cosh\zeta - \sinh\zeta}{\cosh\zeta - \cos\eta}\right) \\
r + r_0 &= \frac{r_0 \sinh\zeta}{\cosh\zeta - \cos\eta} + r_0 = \frac{r_0(\sinh\zeta + \cosh\zeta - \cos\eta)}{\cosh\zeta - \cos\eta} \\
&= r_0\left(\frac{\cosh\zeta + \sinh\zeta}{\cosh\zeta - \cos\eta} - \frac{\cos\eta}{\cosh\zeta - \cos\eta}\right)
\end{aligned}$$

and

$$d_-^2 = 2r_0^2\left(\frac{\cosh\zeta + \sinh\zeta}{\cosh\zeta - \cos\eta}\right) \qquad d_+^2 = 2r_0^2\left(\frac{\cosh\zeta - \sinh\zeta}{\cosh\zeta - \cos\eta}\right)$$

$$d_-^2 + d_+^2 = \frac{4r_0^2\cosh\zeta}{\cosh\zeta - \cos\eta} \qquad d_-^2 - d_+^2 = \frac{4r_0^2\sinh\zeta}{\cosh\zeta - \cos\eta} = 4rr_0$$

$$d_- d_+ = 2r_0^2\sqrt{\left(\frac{\cosh\zeta + \sinh\zeta}{\cosh\zeta - \cos\eta}\right)\left(\frac{\cosh\zeta - \sinh\zeta}{\cosh\zeta - \cos\eta}\right)} = \frac{2r_0^2}{\cosh\zeta - \cos\eta}$$

Moreover, the partial derivatives of the components also become relevant,

$$\begin{aligned}
\partial_r d_- &= \frac{r + r_0}{\sqrt{(r + r_0)^2 + (z - z_0)^2}} = \frac{r + r_0}{d_-} \quad & \partial_r d_+ &= \frac{r - r_0}{\sqrt{(r - r_0)^2 + (z - z_0)^2}} = \frac{r - r_0}{d_+} \\
\partial_z d_- &= \frac{z - z_0}{\sqrt{(r + r_0)^2 + (z - z_0)^2}} = \frac{z - z_0}{d_-} \quad & \partial_z d_+ &= \frac{z - z_0}{\sqrt{(r - r_0)^2 + (z - z_0)^2}} = \frac{z - z_0}{d_+}
\end{aligned}$$

With these formulas established, the first partials of the coordinates can be solved using some heavy application of calculus and the trigonometric/hyperbolic identities $\cos^2\eta + \sin^2\eta = 1$ and $\cosh^2\zeta - \sinh^2\zeta = 1$,

$$\begin{aligned}
\zeta_r &= \partial_r \ln\frac{d_-}{d_+} = \frac{d_+}{d_-}\left(\frac{\partial_r d_-}{d_+} - \frac{d_-\partial_r d_+}{d_+^2}\right) = \frac{\partial_r d_-}{d_-} - \frac{\partial_r d_+}{d_+} = \frac{r+r_0}{d_-^2} - \frac{r-r_0}{d_+^2} \\
&= \frac{1}{2r_0}\left(1 - \frac{\cos\eta}{\cosh\zeta + \sinh\zeta}\right) - \frac{1}{2r_0}\left(\frac{\cos\eta}{\cosh\zeta - \sinh\zeta} - 1\right) \\
&= \frac{1}{2r_0}\left(2 - \cos\eta\left(\frac{1}{\cosh\zeta + \sinh\zeta} + \frac{1}{\cosh\zeta - \sinh\zeta}\right)\right) = \frac{1 - \cosh\zeta\cos\eta}{r_0}
\end{aligned}$$

$$\begin{aligned}
\zeta_z &= \partial_z \ln\frac{d_-}{d_+} = \frac{d_+}{d_-}\left(\frac{\partial_z d_-}{d_+} - \frac{d_-\partial_z d_+}{d_+^2}\right) = \frac{\partial_z d_-}{d_-} - \frac{\partial_z d_+}{d_+} = \frac{z-z_0}{d_-^2} - \frac{z-z_0}{d_+^2} \\
&= \frac{\sin\eta}{2r_0}\left(\frac{1}{\cosh\zeta + \sinh\zeta} - \frac{1}{\cosh\zeta - \sinh\zeta}\right) = \frac{-\sinh\zeta\sin\eta}{r_0}
\end{aligned}$$

$$\begin{aligned}
\eta_r &= \partial_r \arccos\left(\frac{d_-^2 + d_+^2 - 4r_0^2}{2d_-d_+}\right) = \frac{-\partial_r\left(\frac{d_-^2+d_+^2-4r_0^2}{2d_-d_+}\right)}{\sqrt{1 - \left(\frac{d_-^2+d_+^2-4r_0^2}{2d_-d_+}\right)^2}} = \frac{-\partial_r\left(\frac{d_-^2+d_+^2-4r_0^2}{2d_-d_+}\right)}{\sqrt{1-\cos^2\eta}} \\
&= \frac{-1}{\sin\eta}\left(\frac{d_-\partial_r d_- + d_+\partial_r d_+}{d_-d_+} - \frac{d_-^2 + d_+^2 - 4r_0^2}{2d_-^2 d_+^2}(d_+\partial_r d_- + d_-\partial_r d_+)\right) \\
&= \frac{-1}{\sin\eta}\left(\frac{2r}{d_-d_+} - \frac{d_-^2 + d_+^2 - 4r_0^2}{2d_-^2 d_+^2}\left(d_+\frac{r+r_0}{d_-} + d_-\frac{r-r_0}{d_+}\right)\right) \\
&= \frac{-1}{\sin\eta}\left(\frac{2r}{d_-d_+} - \frac{d_-^2 + d_+^2 - 4r_0^2}{2d_-^3 d_+^3}\left(d_+^2(r+r_0) + d_-^2(r-r_0)\right)\right) \\
&= \frac{-1}{\sin\eta}\left(\frac{2r}{d_-d_+} - \frac{1}{2d_-^3 d_+^3}\left(d_-^2 + d_+^2 - 4r_0^2\right)\left((d_+^2 + d_-^2)r + (d_+^2 - d_-^2)r_0\right)\right) \\
&= \frac{-1}{\sin\eta}\left(\frac{2r}{d_-d_+} - \frac{1}{2d_-^3 d_+^3}\left(\frac{4r_0^2\cosh\zeta}{\cosh\zeta - \cos\eta} - 4r_0^2\right)\left(\frac{4r_0^2\cosh\zeta}{\cosh\zeta - \cos\eta}r - 4rr_0^2\right)\right) \\
&= \frac{-1}{\sin\eta}\left(\frac{2r}{d_-d_+} - \frac{8rr_0^4}{d_-^3 d_+^3}\left(\frac{\cosh\zeta}{\cosh\zeta - \cos\eta} - 1\right)^2\right) = \frac{-2r}{d_-d_+\sin\eta}\left(1 - \frac{4r_0^4}{d_-^2 d_+^2}\frac{\cos^2\eta}{(\cosh\zeta - \cos\eta)^2}\right) \\
&= \frac{-2r}{d_-d_+\sin\eta}\left(1 - 4r_0^4\frac{(\cosh\zeta - \cos\eta)^2}{4r_0^4}\frac{\cos^2\eta}{(\cosh\zeta - \cos\eta)^2}\right) = \frac{-2r}{d_-d_+\sin\eta}\left(1 - \cos^2\right) \\
&= \frac{-2r_0\sinh\zeta}{\cosh\zeta - \cos\eta}\frac{\cosh\zeta - \cos\eta}{2r_0^2}\sin\eta = \frac{-\sinh\zeta\sin\eta}{r_0} = \zeta_z
\end{aligned}$$

$$
\begin{aligned}
\eta_z &= \partial_z \arccos\left(\frac{d_-^2 + d_+^2 - 4r_0^2}{2d_-d_+}\right) = \frac{-\partial_z\left(\frac{d_-^2+d_+^2-4r_0^2}{2d_-d_+}\right)}{\sqrt{1-\left(\frac{d_-^2+d_+^2-4r_0^2}{2d_-d_+}\right)^2}} = \frac{-\partial_z\left(\frac{d_-^2+d_+^2-4r_0^2}{2d_-d_+}\right)}{\sqrt{1-\cos^2\eta}} \\
&= \frac{-1}{\sin\eta}\left(\frac{d_-\partial_z d_- + d_+\partial_z d_+}{d_-d_+} - \frac{d_-^2+d_+^2-4r_0^2}{2d_-^2d_+^2}(d_+\partial_z d_- + d_-\partial_z d_+)\right) \\
&= \frac{-1}{\sin\eta}\left(\frac{2(z-z_0)}{d_-d_+} - \frac{d_-^2+d_+^2-4r_0^2}{2d_-^2d_+^2}\left(d_+\frac{z-z_0}{d_-} + d_-\frac{z-z_0}{d_+}\right)\right) \\
&= \frac{-1}{\sin\eta}\left(\frac{2(z-z_0)}{d_-d_+} - \frac{z-z_0}{2d_-^3d_+^3}(d_-^2+d_+^2-4r_0^2)(d_+^2+d_-^2)\right) \\
&= \frac{-2(z-z_0)}{d_-d_+\sin\eta}\left(1 - \frac{1}{4d_-^2d_+^2}\left(\frac{4r_0^2\cosh\zeta}{\cosh\zeta-\cos\eta} - 4r_0^2\right)\frac{4r_0^2\cosh\zeta}{\cosh\zeta-\cos\eta}\right) \\
&= \frac{-2(z-z_0)}{d_-d_+\sin\eta}\left(1 - \frac{4r_0^4}{d_-^2d_+^2}\left(\frac{\cosh\zeta}{\cosh\zeta-\cos\eta} - 1\right)\frac{\cosh\zeta}{\cosh\zeta-\cos\eta}\right) \\
&= \frac{-2(z-z_0)}{d_-d_+\sin\eta}\left(1 - \frac{4r_0^4}{d_-^2d_+^2}\frac{\cosh\zeta\cos\eta}{(\cosh\zeta-\cos\eta)^2}\right) \\
&= \frac{-1}{r_0}\left(1 - 4r_0^4\frac{(\cosh\zeta-\cos\eta)^2}{4r_0^4}\frac{\cosh\zeta\cos\eta}{(\cosh\zeta-\cos\eta)^2}\right) = \frac{\cosh\zeta\cos\eta - 1}{r_0} = -\zeta_r
\end{aligned}
$$

The results give the identities used in the remaining derivations from section 3.1,

$$
\begin{aligned}
\zeta_r &= -\eta_z = \frac{1-\cosh\zeta\cos\eta}{r_0} & \zeta_z &= \eta_r = \frac{-\sinh\zeta\sin\eta}{r_0} \\
\partial_\zeta\zeta_r &= -\partial_\zeta\eta_z = \frac{-\sinh\zeta\cos\eta}{r_0} & \partial_\zeta\zeta_z &= \partial_\zeta\eta_r = \frac{-\cosh\zeta\sin\eta}{r_0} \\
\partial_\eta\zeta_r &= -\partial_\eta\eta_z = \frac{\cosh\zeta\sin\eta}{r_0} & \partial_\eta\zeta_z &= \partial_\eta\eta_r = \frac{-\sinh\zeta\cos\eta}{r_0}
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
\partial_\zeta\zeta_r &= \partial_\eta\zeta_z & \partial_\eta\zeta_r + \partial_\zeta\zeta_z &= 0 \\
\partial_\zeta\eta_z + \partial_\eta\eta_r &= 0 & \partial_\eta\eta_z &= \partial_\zeta\eta_r
\end{aligned}
\tag{3.5}
$$

APPENDIX B: SEPARATION DERIVATIONS

With

$$A = \frac{\cosh\zeta - \cos\eta}{\sinh\zeta}, \qquad\qquad R(\zeta,\eta) = \frac{\sinh\zeta}{\sqrt{\cosh\zeta - \cos\eta}}$$

equation 3.8 is written

$$\Delta^*\psi = \frac{(\cosh\zeta - \cos\eta)^{5/2}}{r_0^3} fg \left[\frac{\partial_\zeta\left(A\partial_\zeta(Rf)\right)}{ARf} + \frac{\partial_\zeta\left(A\partial_\zeta(Rg)\right)}{ARg}\right]$$

where

$$\frac{\partial(A\partial(Rh))}{ARh} = \frac{\partial A\partial R}{AR} + \frac{\partial^2 R}{R} + \frac{h'}{h}\left(\frac{\partial A}{A} + 2\frac{\partial R}{R}\right) + \frac{h''}{h}$$

with an arbitrary function h (of the same single variable as the associated partial) standing in for f and g respectively.

Then

$$\frac{\partial_\zeta A\partial_\zeta R}{AR} + \frac{\partial_\zeta^2 R}{R} + \frac{\partial_\eta A\partial_\eta R}{AR} + \frac{\partial_\eta^2 R}{R} = f^*(\zeta) + g^*(\eta) \qquad \text{(3.10 Revisited)}$$

and

$$\frac{\partial_\zeta A}{A} + 2\frac{\partial_\zeta R}{R} = f^{**}(\zeta) \qquad \frac{\partial_\eta A}{A} + 2\frac{\partial_\eta R}{R} = g^{**}(\eta) \qquad \text{(3.2 Revsited)}$$

To compute the unknown terms, start by finding $f^{**}(\zeta)$, where the components are

$$\begin{aligned}
\partial_\zeta A(\zeta,\eta) &= \frac{\sinh^2\zeta - (\cosh\zeta - \cos\eta)\cosh\zeta}{\sinh^2\zeta} = \frac{\cosh\zeta\cos\eta - 1}{\sinh^2\zeta} \\
\frac{\partial_\zeta A}{A} &= \frac{\cosh\zeta\cos\eta - 1}{(\cosh\zeta - \cos\eta)\sinh\zeta}
\end{aligned}$$

and

$$\begin{aligned}
\partial_\zeta R(\zeta,\eta) &= \frac{\cosh\zeta}{\sqrt{\cosh\zeta - \cos\eta}} - \frac{\sinh^2\zeta}{2(\cosh\zeta - \cos\eta)^{3/2}} \\
&= \frac{2(\cosh\zeta - \cos\eta)\cosh\zeta - \sinh^2\zeta}{2(\cosh\zeta - \cos\eta)^{3/2}} \\
&= \frac{\cosh^2\zeta - 2\cosh\zeta\cos\eta + 1}{2(\cosh\zeta - \cos\eta)^{3/2}} \\
\frac{\partial_\zeta R}{R} &= \frac{\cosh^2\zeta - 2\cosh\zeta\cos\eta + 1}{2(\cosh\zeta - \cos\eta)\sinh\zeta}
\end{aligned}$$

Then

$$f^{**}(\zeta) = \frac{\partial_\zeta A}{A} + 2\frac{\partial_\zeta R}{R} = \frac{\cosh^2\zeta - \cosh\zeta\cos\eta}{(\cosh\zeta - \cos\eta)\sinh\zeta} = \frac{\cosh\zeta}{\sinh\zeta}$$

which is indeed a function of a single variable.

Similarly, for $g^{**}(\eta)$,

$$\begin{aligned}
\partial_\eta A(\zeta,\eta) &= \frac{\sin\eta}{\sinh\zeta} & \partial_\eta R(\zeta,\eta) &= \frac{-\sinh\zeta\sin\eta}{2(\cosh\zeta - \cos\eta)^{3/2}} \\
\frac{\partial_\eta A}{A} &= \frac{\sin\eta}{\cosh\zeta - \cos\eta} & \frac{\partial_\eta R}{R} &= \frac{-\sin\eta}{2(\cosh\zeta - \cos\eta)}
\end{aligned}$$

so that

$$g^{**}(\eta) = \frac{\partial_\eta A}{A} + 2\frac{\partial_\eta R}{R} = 0$$

Using these results condenses equation 3.8,

$$\Delta^* \psi = \frac{(\cosh\zeta - \cos\eta)^{5/2}}{r_0^3} fg \left[f^* + \frac{f' \cosh\zeta}{f \sinh\zeta} + \frac{f''}{f} + g^* + \frac{g''}{g} \right]$$

and for the other unknown terms, revisit equation 3.10

$$\frac{\partial_\zeta A \partial_\zeta R}{AR} + \frac{\partial_\zeta^2 R}{R} + \frac{\partial_\eta A \partial_\eta R}{AR} + \frac{\partial_\eta^2 R}{R} = f^*(\zeta) + g^*(\eta) \qquad \text{(3.10 Revisited)}$$

proceeding to compute each of the component summands.

Using the results from above,

$$\begin{aligned}
\frac{\partial_\zeta A \partial_\zeta R}{AR} &= \left(\frac{\cosh\zeta \cos\eta - 1}{(\cosh\zeta - \cos\eta)\sinh\zeta} \right) \left(\frac{\cosh^2\zeta - 2\cosh\zeta\cos\eta + 1}{2(\cosh\zeta - \cos\eta)\sinh\zeta} \right) \\
\frac{\partial_\eta A \partial_\eta R}{AR} &= \frac{-\sin^2\eta}{2(\cosh\zeta - \cos\eta)^2}
\end{aligned}$$

Since the first term is more involved, its numerator is expanded separately,

$$\begin{aligned}
&(\cosh\zeta\cos\eta - 1)(\cosh^2\zeta - 2\cosh\zeta\cos\eta + 1) \\
&\quad = \left(\cosh\zeta\cos\eta - \cosh^2\zeta + \sinh^2\zeta\right)\left((\cosh\zeta - \cos\eta)^2 + \sin^2\eta\right) \\
&\quad = (\cos\eta - \cosh\zeta)\left((\cosh\zeta - \cos\eta)^2 + \sin^2\eta\right)\cosh\zeta \\
&\qquad + (\cosh\zeta - \cos\eta)^2\sinh^2\zeta + \sinh^2\zeta\sin^2\eta \\
&\quad = (\cosh\zeta - \cos\eta)^2\sinh^2\zeta + \sinh^2\zeta\sin^2\eta \\
&\qquad - (\cosh\zeta - \cos\eta)\left((\cosh\zeta - \cos\eta)^2 + \sin^2\eta\right)\cosh\zeta
\end{aligned}$$

so that

$$\frac{\partial_\zeta A \partial_\zeta R}{AR} = \frac{1}{2} + \frac{\sin^2\eta}{2(\cosh\zeta - \cos\eta)^2} - \frac{\big((\cosh\zeta - \cos\eta)^2 + \sin^2\eta\big)\cosh\zeta}{2(\cosh\zeta - \cos\eta)\sinh^2\zeta}$$

and taken one step further,

$$\begin{aligned}(\cosh\zeta - \cos\eta)^2 + \sin^2\eta &= \cosh^2\zeta - 2\cosh\zeta\cos\eta + 1\\ &= 2\cosh^2\zeta - 2\cosh\zeta\cos\eta - \sinh^2\zeta\\ &= 2\cosh\zeta(\cosh\zeta - \cos\eta) - \sinh^2\zeta\end{aligned}$$

yielding a more useful form of the above,

$$\begin{aligned}\frac{\partial_\zeta A \partial_\zeta R}{AR} &= \frac{1}{2} + \frac{\sin^2\eta}{2(\cosh\zeta - \cos\eta)^2} - \frac{\big(2\cosh\zeta(\cosh\zeta - \cos\eta) - \sinh^2\zeta\big)\cosh\zeta}{2(\cosh\zeta - \cos\eta)\sinh^2\zeta}\\ &= \frac{1}{2} + \frac{\sin^2\eta}{2(\cosh\zeta - \cos\eta)^2} - \frac{\cosh^2\zeta}{\sinh^2\zeta} + \frac{\cosh\zeta}{2(\cosh\zeta - \cos\eta)}\end{aligned}$$

The combined partial sum then becomes,

$$\frac{\partial_\zeta A \partial_\zeta R}{AR} + \frac{\partial_\zeta A \partial_\zeta R}{AR} = \frac{1}{2} - \frac{\cosh^2\zeta}{\sinh^2\zeta} + \frac{\cosh\zeta}{2(\cosh\zeta - \cos\eta)}$$

For the remaining terms of equation 3.10, one more of each partial derivative must be taken for R,

$$\begin{aligned}
\partial_\zeta^2 R &= \partial_\zeta \left(\frac{\cosh^2\zeta - 2\cosh\zeta\cos\eta + 1}{2(\cosh\zeta - \cos\eta)^{3/2}} \right) \\
&= \frac{(\cosh\zeta - \cos\eta)\sinh\zeta}{(\cosh\zeta - \cos\eta)^{3/2}} - \frac{3(\cosh^2\zeta - 2\cosh\zeta\cos\eta + 1)\sinh\zeta}{4(\cosh\zeta - \cos\eta)^{5/2}} \\
&= \frac{\sinh\zeta}{4(\cosh\zeta - \cos\eta)^{5/2}} \left(4(\cosh\zeta - \cos\eta)^2 - 3\cosh^2\zeta + 6\cosh\zeta\cos\eta - 3\right) \\
&= \frac{\sinh\zeta}{4(\cosh\zeta - \cos\eta)^{5/2}} \left(\cosh^2\zeta - 2\cosh\zeta\cos\eta + 4\cos^2\eta - 3\right) \\
&= \frac{\sinh\zeta}{4(\cosh\zeta - \cos\eta)^{5/2}} \left((\cosh\zeta - \cos\eta)^2 - 3\sin^2\eta\right)
\end{aligned}$$

$$\begin{aligned}
\partial_\eta^2 R &= \partial_\eta \left(\frac{-\sinh\zeta\sin\eta}{2(\cosh\zeta - \cos\eta)^{3/2}} \right) \\
&= \frac{-\sinh\zeta\cos\eta}{2(\cosh\zeta - \cos\eta)^{3/2}} + \frac{3\sinh\zeta\sin^2\eta}{4(\cosh\zeta - \cos\eta)^{5/2}} \\
&= \frac{\sinh\zeta}{4(\cosh\zeta - \cos\eta)^{5/2}} \left(3\sin^2\eta - 2(\cosh\zeta - \cos\eta)\cos\eta\right)
\end{aligned}$$

so that together

$$\begin{aligned}
\partial_\zeta^2 R + \partial_\eta^2 R &= \frac{\sinh\zeta}{4(\cosh\zeta - \cos\eta)^{5/2}} \left((\cosh\zeta - \cos\eta)^2 - 2(\cosh\zeta - \cos\eta)\cos\eta\right) \\
&= \frac{(\cosh\zeta - 3\cos\eta)\sinh\zeta}{4(\cosh\zeta - \cos\eta)^{3/2}}
\end{aligned}$$

$$\frac{\partial_\zeta^2 R + \partial_\eta^2 R}{R} = \frac{\cosh\zeta - 3\cos\eta}{4(\cosh\zeta - \cos\eta)} = \frac{1}{4} - \frac{\cos\eta}{2(\cosh\zeta - \cos\eta)}$$

Combining the results,

$$
\begin{aligned}
f^*(\zeta) + g^*(\eta) &= \frac{\partial_\zeta A \partial_\zeta R}{AR} + \frac{\partial_\eta A \partial_\eta R}{AR} + \frac{\partial_\zeta^2 R}{R} + \frac{\partial_\eta^2 R}{R} \\
&= \frac{3}{4} - \frac{\cosh^2 \zeta}{\sinh^2 \zeta} + \frac{\cosh \zeta}{2(\cosh \zeta - \cos \eta)} - \frac{\cos \eta}{2(\cosh \zeta - \cos \eta)} \\
&= \frac{5}{4} - \frac{\cosh^2 \zeta}{\sinh^2 \zeta} = \frac{1}{4} - \frac{1}{\sinh^2 \zeta}
\end{aligned}
$$

showing that with the separation, equation 3.8 can be written in total,

$$
\Delta^* \psi = \frac{(\cosh \zeta - \cos \eta)^{5/2}}{r_0^3} fg \left[\frac{1}{4} - \frac{1}{\sinh^2 \zeta} + \frac{f' \cosh \zeta}{f \sinh \zeta} + \frac{f''}{f} + \frac{g''}{g} \right]
$$