GEOMETRIC SPINORS

by

Jordan Gemmill

THESIS

Presented to the Faculty of

The University of Houston-Clear Lake

In Partial Fulfillment

Of the Requirements

For the Degree

MASTER OF SCIENCE

in Physics

THE UNIVERSITY OF HOUSTON-CLEAR LAKE

AUGUST, 2021

GEOMETRIC SPINORS

by

Jordan Gemmill

APPROVED BY

Samina Masood, PhD, Chair

David Garrison, PhD, Committee Member

Paul Withey, PhD, Committee Member

RECEIVED/APPROVED BY THE COLLEGE OF SCIENCE AND ENGINEERING:

David Garisson, PhD, Interim Associate Dean

Miguel Gonzales, PhD, Dean

Dr. Van Eric Mayes 1978-2021

Dedication

This thesis is dedicated to my good friend and mentor, Dr. Mayes.

Thank you, for not only teaching me how to do physics, but for teaching me everything I needed to be a physicists.

Acknowledgments

First and foremost I would like to thank my lovely fiancé, Victoria, for without her support, love, and unending patience, I would not have made it this far. I would like to acknowledge my thesis advisor Dr. Masood, not just in all of her support in completing this thesis, but for all of the support that she has given me since my arrival here at UHCL. I am truly inspired by her devotion to students and her ability to foster a warm, supportive community within the program. I would also like to express my deepest gratitude for my committee members Dr. David Garrison and Dr. Paul Withey. To Dr. Garrison, thank you for always challenging me. I am especially grateful for the computational senior project I undertook with you in the final year of my undergraduate degree. Without you pushing me, I would not have gain the vital skills necessary for undertaking such a large project. To Dr. Withey, thank you for always sharing your passion, for the many interesting conversations, and for all the guidance you have given me. I would also like to thank my good friend and mentor, Dr. Alan Weiderhold. Without the many hours you have invested in me, and the many hours you have invested into correcting my chalk-board expressions, I would not have developed the skills so vital to my success. I would also like to blame Dr. Weiderhold for initially teaching me Clifford algebra, as he is partially responsible for all the unsolicited geometric algebra conversations that I have thrust upon the good people of UHCL's stem building. I am also grateful for the many books he has given me, especially the one he unknowingly gifted me. I am forever indebted to my friend and fellow student, David Ortega. Without his support and encouragement, I would have never had the confidence to reach out to those who I admire most. I would like to thank Dr. Doran for kindly inviting me to the Clifford Dev. Slack group, which has greatly contributed to my understanding of geometric algebra by allowing me to leach from so many great minds. I would also like to thank Dr. Hestenes for taking the time to trade incredible, serendipitous stories with me. His 1966 book Space-Time Algebra magically appeared on my bookshelf (unknowingly gifted to me by Dr. Weiderhold). I am also blown away by Dr. Hestenes' kind introduction to Chapman University's Dr. Justin Dressel and Luke Barnes. I would like to thank Dr. Dressel for his detailed and enlightening correspondence, which was so interesting I had to refrain from reading them at times, lest I be sent down another avenue of research at a time I ought to be writing to finish this work. I am also grateful for Craig Brooks for sharing his experience in navigating the thesis process, and for his help with troubleshooting LaTeX at certain points in the process. I would also like to thank Jonathan Miller for always being invested in my success and for allowing me to be invested in his. I must also acknowledge my long time friend Michael Craig, whose random texts of encouragement always seemed to find me when I needed them the most. And of course, I would like to thank my Mom and Dad for their endless love, support and encouragement. I am also grateful for my soon-to-be mother and father-in-law for the countless times and ways in which they have helped me throughout this process. I am also grateful for my siblings who continually lift me up, and to all my family who have reached out to support and champion me. Finally, I would like to acknowledge all of the staff, whose hard work, makes UHCL's day-to-day operations possible. I am especially thankful for Maria Elena. She is most certainly the force which binds the collective efforts of the department.

ABSTRACT

GEOMETRIC SPINORS

Jordan Gemmill

University of Houston-Clear Lake, 2021

Thesis Chair: Samina Masood, PhD

Geometric Algebra is a unique variant of what is otherwise known as Clifford Algebra. In this work we show that the geometric algebra provides better tools to visualize physical problems, benefited by our natural geometric intuition. Geometric algebra provides a routine and systematic way to analyze physical systems. It is demonstrated that the calculations of magnetic moment with constant magnetic field and that of the oscillating magnetic field, can both be expressed in a single expression. Using the geometric algebra we reproduce the solutions of Schrodinger's equation in quantum mechanics, and show that the spacetime algebra can express Dirac's equation without the use of imaginary numbers or matrices.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

Geometric algebra is a mathematical approach to Clifford algebra that has not yet been fully appreciated. Often, geometric algebra is viewed as Clifford algebra but by another name. From a purely mathematical standpoint this could be argued. After all, the equivalence between Hestenes' geometric algebra and the universal Clifford algebra of signature (p, q) has already been proven[2]. That is two mathematical systems, with two different sets of axioms, can be shown to be equivalent. However, the approaches are markedly different.

In [2], the authors state that a key difference between the two approaches, lies with the bilinear form used in their construction. In the Clifford algebra approach discussed in [2], a bilinear form is assumed *a priori*, and it remains static. Whereas, in the geometric algebra approach, the bilinear form is obtained from the geometric product, or as they state *a posteriori*. Which, in this case means that the bilinear form can be modified by choosing a different geometric product. The bilinear form being discussed here is of course

$$\mathcal{B}(x, y) = x \cdot y, \tag{1.1}$$

the familiar dot product. In the Clifford approach the bilinear form is assumed. In Hestenes' construction, it is defined in terms of the more fundamental geometric product

$$a \cdot b = \frac{1}{2}(ab + ba), \tag{1.2}$$

where ab is the geometric product. As trivial as this may appear at first glance, it has profound implications concerning the conceptual and technical hurdles practitioners must over come before implementing such a system.

The authors of [2] argue that Hestenes' approach is full of geometric significance, free of any basis, and that it only requires basic concepts of linear algebra. Contrast this with the background needed to approach Clifford algebra in its more traditional presentations, which usually require advanced mathematical concepts just to state definitions. These definitions are important for those interested in pure mathematics, but for those interested in applications to physics and engineering, they serve only as an unnecessary barrier.

1.1 The Development of Geometric Algebra

In this work, we introduce geometric algebra in detail and demonstrate its relevance for calculating physical quantities in an effective way. We show that we can use geometric algebra to help develop our understanding of physical systems, in a way that is benefited by our natural geometric intuition.

Clifford Algebra begins with Hermann Grassmann. Grassmann had more to do with the creation of Clifford algebra than Clifford did himself. Grassmann, was a German school teacher whose mathematical works garnered little attention in his own time, so much so that he eventually stepped away from the field of mathematics altogether. In 1844 Grassmann published his most important work, *Die lineale Ausdehnungslehre* (Linear Extension Theory). The work was suffered one major problem, it was too far ahead of its time. Grassman's genius was an island unto its own. Within Ausdehnungslehre, Grassmann laid out the ideas and notions for vector spaces, span, basis, subspaces, dimensions, join and meet of subspaces, and the projection of onto these subspaces [9]. He even obtains the formula for a change of coordinates under a change of basis, all at a time before the language even existed to speak of such ideas. In fact rigorous definitions for many of his ideas, such as vector spaces, did not appear until the 1920's, when Hermann Weyl and others work gave formal definitions for such ideas [9]. It is then little wonder why his work went mostly unappreciated in his lifetime. Of the few well known mathematicians of Grassmann's day who did come into contact with his work, nearly all failed to appreciate its content. The well know German mathematician Möbius deemed Grassmann's work unreadable. Hamilton, Irish mathematician and physicists, wrote to De Morgan that to read Grassmann's work he would have to take up smoking, some rather harsh words [5]. However, not everyone of the time shared these views.

Figure 1.1: *"I remain completely confident that the labour I have expended on the science presented here and which has demanded a significant part of my life as well as the most strenuous application of my powers, will not be lost. It is true that I am aware that the form which I have given the science is imperfect and must be imperfect. But I know and feel obliged to state (though I run the risk of seeming arrogant) that even if this work should again remain unused for another seventeen years or even longer, without entering into the actual development of science, still that time will come when it will be brought forth from the dust of oblivion and when ideas now dormant will bring forth fruit. I know that if I also fail to gather around me (as I have until now desired in vain) a circle of scholars, whom I could fructify with these ideas, and whom I could stimulate to develop and enrich them further, yet there will come a time when these ideas, perhaps in a new form, will arise anew and will enter into a living communication with contemporary developments. For truth is eternal and divine." -Hermann Grassmann*

(preface to the 1862 2nd Edition of Ausdehnungslehre as translated in [5])
Attribution: Public Domain Source:Grassmann Portrait

Clifford, an English mathematician and philosopher, was one of the few of his day to be both aware of Grassmann's work and of Hamilton's work on the quaternions [5]. Clifford, its seems, possessed an almost supernatural ability to anticipate the future. In 1870, Clifford submitted an abstract to the Proceedings of the Cambridge Philosophical Society, in which he makes bold speculation about the nature of reality as it relates to geometry, anticipating Einstein's general relativity, as pointed out in [12]. Another instance his vision, can be seen in his praise of Grassmann's work. A quote from Clifford's 1878 work is given in figure 1.2. In that same work, Clifford goes on to show how Hamilton's quaternions fit within Grassmann's theory of extension, uniting the two, calling the union of their systems geometric algebra [4].

Unfortunately, Clifford's life was cut short the following year due to tuberculosis, at the age of 33 and was unable to advance the system he helped create. In the years following his death, mathematicians categorized and classified, what they now called, Clifford Algebras, stripping away the link between Grassmann and Clifford. Who knows how the system might have developed with Clifford at the helm?

What did happen is geometric algebra, and Hamilton's quaternions, faded into the background, being eclipsed by the vectorial system developed by English physicist Oliver Heaviside and American physicist Josiah Willard Gibbs.

Physicists began employing Clifford algebras just before the 1930's, when Pauli, who was incorporating spin into Schrodinger's matrix mechanics, found it necessary to introduce his famous spin matrices, and subsequently, when Dirac introduced its relativistic extension.

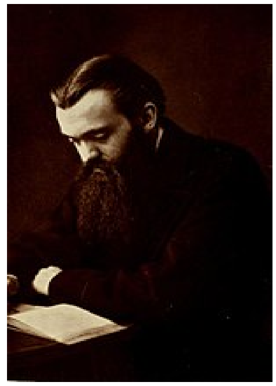

Figure 1.2: *"Until recently I was unacquainted with the Ausdehnungslehre, and knew only so much of it as is contained in the author's geometrical papers in Crelle's Journal and in Hankel's Lectures on Complex Numbers. I may, perhaps, therefore be permitted to express my profound admiration of that extraordinary work, and my conviction that its principles will exercise a vast influence upon the future of mathematical science." - William Clifford*

(Introductory paragraph of Clifford's 1878 work Applications of Grassmann's Extensive Algebra [4])
Attribution: Public Domain   Source:Clifford Portrait

1.2 Geometric Algebra

In this section we introduce some necessary terminology and operations needed to develop geometric algebra, which will be crucial for understanding the next section, where we define the geometric product of two vectors.

Geometric algebras are graded algebras, meaning they contain elements of different grades. The elements of a geometric algebra are called *multivectors*. Every multivector is a distinct sum of *k-vectors*. A k-vector is also referred to as a multivector of *grade-k*. Every k-vector is a sum of *k-blades*, and finally, a k-blade is any multivector $\mathbf{A}_k$ that can be factored into a product of k anticommuting vectors, such as

$$\mathbf{A}_k = \mathbf{a}_1\mathbf{a}_2...\mathbf{a}_k, \quad \text{with} \quad \mathbf{a}_i\mathbf{a}_j = -\mathbf{a}_j\mathbf{a}_i \quad \text{for} \quad i \neq j. \tag{1.3}$$

This can be very confusing in the beginning and it is difficult to see the need for both of the terms k-vectors and k-blades. However, the distinction is an important one because geometric algebra represents subspaces with blades. For instance, when working in the geometric algebra Blades are the most basic building blocks of geometric algebra. They represent the different subspaces that together make up a geometric algebra. Blades allow us to represent points, lines, planes, volumes, etc., as a product of vectors. For instance, in $\mathcal{G}_3$ a 3-blade $(\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3)$ represents an oriented volume. A scalar is represented by a product of zero vectors.

A general multivector can be represented as a unique sum of its k-vector constituents. As an analogy, take an arbitrary quadratic polynomial

$$f(x) = a_0 + a_1 x + a_2 x^2. \tag{1.4}$$

We say that $f(x)$ is a polynomial function of degree-2, and that it is a sum of degree-0, degree-1, and degree-2 *independent* terms. Written as as sum, $f(x)$ takes the form

$$f(x) = \sum_{n=0}^{2} a_n x^n. \tag{1.5}$$

Similarly, any multivector may be written as a sum of k-vectors

$$M = M_0 + M_1 + M_2 + \ldots + M_n = \sum_{k=0}^{n} \langle M \rangle_k, \tag{1.6}$$

where we have introduced the notation $\langle\ \rangle_k$ for the *grade-projection operator*. Application of the grade-k projection operator to a multivector M selects out the k-vector term in M as such

$$\langle M \rangle_k = \left\langle \sum_{n=0} \langle M \rangle_n \right\rangle_k = \langle M_0 + M_1 + \ldots M_k + \ldots \rangle_k \tag{1.7}$$

$$\langle M \rangle_k = \langle M \rangle_{n=k} = \langle M_k \rangle_k = M_k \tag{1.8}$$

Thus we can say that the grade-k projection operator applied to the multivector M is equal to the *k-vector* term in M or

$$\langle M \rangle_k = M_k. \tag{1.9}$$

Polynomials are often referred to as being constant, linear, quadratic, etc. Often times though, we instead refer to polynomials by their degree, with it being understood that a constant is zeroth degree polynomial, a first degree polynomial is linear, and so on. In much the same way, a 0-vector is called a scalar, a 1-vector, simply vector, a 2-vector is called a bivector, and so on. Every element of $\mathcal{G}$ is a multivector.

To make things more concrete, we give a few examples of multivectors, in different forms, from different geometric algebras. A general multivector M belonging to the space $\mathcal{G}_3$ can be written as

$$M = s + \mathbf{v} + \mathbf{B} + \mathbf{T}. \tag{1.10}$$

Where, s is a scalar, $\mathbf{v}$ is a vector, $\mathbf{B}$ is a bivector, and $\mathbf{T}$ is a trivector. The table below recaps this notational information.

Object	**Grade**	**Symbol**	**Notation**
Multivector	All	M	uppercase Latin
scalar	0	s	lowercase Latin
vector	1	**v**	lowercase Latin,bold
Bivector	2	**B**	uppercase Latin, bold
Trivector	3	**T**	uppercase Latin, bold

Table 1.1: *Terminology and Notations for* $M \in \mathcal{G}_3$

In general, a geometric algebra is denoted by $\mathcal{G}(p, q)$, where the *signature* $\{p, q\}$ denotes the number of basis vectors that generate the space. The number of basis vectors which have positive square are given by p, and the number of basis vectors which have negative square are given by q. When working with geometric algebras in which $q = 0$, we denote $\mathcal{G}(p, 0)$ by the shorthand $\mathcal{G}_p$.

1.3 Geometric Product of Vectors

We now develop the fundamental product of geometic algebra, the geometric product of vectors. The geometric product of two vectors is foundational because from this simple product, we may construct the entire algebra.

Starting with a vector space $\mathcal{V}_n$ over the field of real numbers $\mathbb{R}$, where $\mathbf{a}, \mathbf{b}, \mathbf{c}... \in \mathcal{V}$, the *geometric product* for vectors is defined by the following properties:

$$(\mathbf{ab})\mathbf{c} = \mathbf{a}(\mathbf{bc}) \qquad \text{associative} \tag{1.11}$$

$$\mathbf{a}(\mathbf{b}+\mathbf{c}) = \mathbf{ab}+\mathbf{ac} \qquad \text{left distributive} \tag{1.12}$$

$$(\mathbf{b}+\mathbf{c})\mathbf{a} = \mathbf{ba}+\mathbf{ca} \qquad \text{right distributive} \tag{1.13}$$

$$\mathbf{a}^2 = \mathbf{aa} = a^2. \qquad \text{contraction} \tag{1.14}$$

The first property, associativity, is all but too familiar. Properties, two and three, must be specified as the geometric product is not commutative. These first three properties taken together, are rather mundane, no different than the rules defining matrix multiplication, which is associative and also lacks commutativity, but the final property, contraction, makes geometric algebra distinct from every other associative algebra, and simple as it may seem, it is the source of geometric algebra's rich structure. An immediate consequence is the existence of multiplicative inverses for every non-zero vector. This allows to divide by vectors, a property that greatly aid and facilitates computations.

We can determine a vectors inverse by starting with,

$$\mathbf{r}^{-1}\mathbf{r} = 1 \tag{1.15}$$

right multiplying by $\mathbf{r}$ gives

$$\mathbf{r}^{-1} = \frac{\mathbf{r}}{r^2}, \tag{1.16}$$

after applying the contraction property and dividing by the result. Notice that $\mathbf{r}^{-1}\mathbf{r} = \mathbf{r}\mathbf{r}^{\mathbf{-1}} = \mathbf{1}$, that is the geometric product of any vector and it's inverse is commutative. While the geometric product is not defined to be commutative, it is better to think of the product as not necessarily commutative. This is a special case of the following, more general result.

Consider the case where $\mathbf{a}$ and $\mathbf{b}$ are parallel to one another. Recall, two vectors are said to be parallel if one can be written as a scalar multiple of the other i.e. $\mathbf{b} = c\mathbf{a}$. In this case

$$\mathbf{ab} = \mathbf{a}(c\mathbf{a}) = c\mathbf{a}(\mathbf{a}) = \mathbf{ba}, \tag{1.17}$$

and so the product of parallel vectors is commutative, and therefore a vector and it's inverse are parallel. In the opposite extreme, consider the vector equation

$$\mathbf{a} + \mathbf{b} = \mathbf{c} \tag{1.18}$$

where $\mathbf{a}$ and $\mathbf{b}$ are orthogonal. This is a vector equation defining a right triangle. Squaring both sides of (1.18) results in

$$\mathbf{a}^2 + \mathbf{ab} + \mathbf{ba} + \mathbf{b}^2 = \mathbf{c}^2. \tag{1.19}$$

By (1.14), the square of every vector must reduce to a real scalar. Therefore,

$$a^2 + b^2 = c^2, \tag{1.20}$$

we recognize the only way can satisfied is if

$$\mathbf{ab} + \mathbf{ba} = 0, \tag{1.21}$$

the geometric product of orthogonal vectors is anticommutative

$$\mathbf{ab} = -\mathbf{ba}. \tag{1.22}$$

We have now established the two most important special case for geometric multiplication. Since, in general the product of vectors is neither commutative nor anticommutativ, but using these two special cases we can always split a general product into two terms, a parallel term and an orthogonal term. The loss of commutativity may seem like a burden, but it is actually a great source of strength.

The fundamental identity of geometric algebra is

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}. \tag{1.23}$$

The geometric product, when it takes two vectors as inputs, returns a scalar and a *bivector*. The scalar portion is the symmetric term, and is a measure of how colinear the vectors are. The bivector term is a new object, and is the antisymmetric portion of the product. Bivectors, like vectors, posses both magnitude and orientation. They may be interpreted as oriented plane segment, with the orientation coinciding with rotating **a** onto **b**. The symmetric portion is called *inner product*, or "dot" product, and the antisymmetric term is called *outer product* or "wedge" product. These products are defined by the geometric product itself, a feature unique to geometric algebra. This has several advantages. First both of these products are geometrically motivated, meaning that they have physical consequences when applied in real space. Second, by defining these products with the more fundamental geometric product, we will be able to obtain important results much easier than if we worked with two separate products. Examples of this will demonstrated in the sections covering reflections and rotations. The inner product for two vectors is defined by

$$\mathbf{a} \cdot \mathbf{b} = \frac{1}{2}(\mathbf{ab} + \mathbf{ba}), \tag{1.24}$$

and similarly the outer product is be defined as

$$\mathbf{a} \wedge \mathbf{b} = \frac{1}{2}(\mathbf{ab} - \mathbf{ba}). \tag{1.25}$$

The results of this section apply to geometric algebras of every dimension and signature. In the following section we introduce an important multivector, the pseudoscalar. It is similar to a volume-form, from the formalism of differential forms.

1.3.1 Pseudoscalar

Pseudoscalars, by definition, are grade-n multivectors belonging to $\mathcal{G}_n$, and are therefore the highest grade objects within the algebra. This gives the pseudoscalar some rather interesting algebraic and geometric properties, regardless of the specific geometric algebra. For instance, for any vector $\mathbf{a}$, its geometric product with the pseudoscalar

$$\mathbf{aI} = \mathbf{a} \cdot \mathbf{I}, \tag{1.26}$$

always results in an $n-1$ vector, since $\mathbf{a} \wedge \mathbf{I}$ is necessarily zero. This will be an important fact when we discuss duality. Also, this facet of the pseudoscalar is not limited to its interactions with vectors, for any k-blade $\mathbf{A}_k$ we have

$$\mathbf{A}_k\mathbf{I} = \langle \mathbf{A}_k\mathbf{I} \rangle_{n-k}, \tag{1.27}$$

where $1 \leq k \leq n$. The reason grade-n multivectors are termed pseudoscalars comes from their commutation properties when multiplying other elements of the algebra. Depending on whether n is even or odd, pseudoscalars are imbued with different commutation properties. When n is odd, the pseudoscalar commutes with all grade-1 multivectors, and by extension every multivector belonging to the space [10]. This is especially important in $\mathcal{G}_3$ where we take advantage of the fact that $\mathbf{I}$ commutes with all elements of the algebra, and so behaves like a usual scalar. When n is even however, the pseudoscalar anticommutes with all odd grade multivectors, and so the term pseudoscalar is fitting because when n is odd it commutes with all multivectors and so behaves like a scalar. But when n is even it anticommutes with all odd multivectors. Whether, n is even or odd the pseudoscalar commutes with all even grade multivectors [6]. These relations are all summarized by

$$\mathbf{I}A_k = (-1)^{k(n-1)} A_k \mathbf{I} \tag{1.28}$$

Other suitable names for $\mathbf{I}$ are volume-form or oriented n-volume, but the term pseudoscalar is preferred within geometric algebra, as it emphasizes the fact that depending on the dimension of the space we are working with, it will either behave entirely like a scalar, or it may behave like an additional anticommuting vector for the odd grade elements.

In the next chapter we show how to construct a canonical basis for $\mathcal{G}_2$ and $\mathcal{G}_3$.

CHAPTER 2

GEOMETRIC ALGEBRA

We introduce two specific geometric algebras, $\mathcal{G}(2,0)$ and $\mathcal{G}(3,0)$. Their relation to other mathematical structures and how geometric algebra encompassed and extends these systems is shown as well.

2.1 Plane Geometric Algebra

Before extending the Euclidean basis $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2\}$ to the canonical basis of $\mathcal{G}_2$, we note several interesting propertiers of $\mathcal{G}_2$ before moving on and demonstrating these properties with a basis. The basis free form of a general multivector belonging to $\mathcal{G}_2$ has the form

$$A = a + \mathbf{v} + \mathbf{B}, \tag{2.1}$$

and as we will demonstrate in the following sections, this can also be represented as

$$A = a + \mathbf{v} + \mathbf{i}b. \tag{2.2}$$

We have introduced a single exception to the notation used throughout this work, but it is well motivated. That is, it is customary to represent the pseudoscalar of the plane, a bivector, with $\mathbf{i}$. This is done intentionally because is behaves almost identically to the imaginary scalar i, from the complex plane. However, we should not forget that geometric algebra is defined to be an extension of the real numbers to incorporate oriented subspaces. There is no need to introduce complex scalars as their is plenty of complex structure within the real geometric. This complex structure is given by higher grade objects that square to -1. By working with only real scalars, we are able to maintain clear geometric interpretations of the objects involved, this is done by identifying blades with geometric primitives, and including uninterpreted imaginary scalars would only lead to obscuring the geometric picture provided by the blades.

There is an important structure nested within $\mathcal{G}_2$. This substructure is technically an algebra all on its own, and this algebra is identical to the algebra of the complex numbers. Notice that if we only take the even grade multivectors of the above equation we have

$$Z = a + \mathbf{i}b. \tag{2.3}$$

Because Z is a sum of even grade elements, we say it belongs to the even subalgebra of $\mathcal{G}_2$, which is denoted by $\mathcal{G}_2^+$, and so it is not uncommon to find expressions like

$$A = \mathbf{v} + Z, \tag{2.4}$$

for a general multivector in $\mathcal{G}_2$, especially as this form calls attention the even and odd grading of the multivector. In terms of a basis this is written as

$$A = a + v_1\boldsymbol{\sigma}_1 + v_2\boldsymbol{\sigma}_2 + b\boldsymbol{\sigma}_1 \wedge \boldsymbol{\sigma}_2. \tag{2.5}$$

We now demonstrate how a canonical basis for $\mathcal{G}_2$ can be constructed by starting with an orthonormal basis for the Euclidean vector space $\mathbb{R}^2$. These basis vectors satisfy,

$$\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_i = 1 \qquad \text{for } i = 1, 2 \tag{2.6}$$

$$\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j = 0 \qquad \text{for } i \neq j \tag{2.7}$$

by definition since they are assumed orthonormal. We now take all possible combinations of geometric products between basis vectors, such that the result is linearly independent from any other basis element. The total number of basis elements in any geometric algebra is always 2^n, where n is the dimension of the generating vector space. Therefore, we expect to generate two additional basis elements. Similar to (2.6), the geometric product allows us to write

$$\boldsymbol{\sigma}_i^2 = 1 \qquad \text{for } i = 1, 2. \tag{2.8}$$

Notice the lack of the dot product, only the contraction property has been used, generating a scalar. Taking the only independent product left, we obtain

$$\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + \boldsymbol{\sigma}_1 \wedge \boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_1 \wedge \boldsymbol{\sigma}_2. \tag{2.9}$$

The inner product is zero since $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are orthogonal. The term $\boldsymbol{\sigma}_1 \wedge \boldsymbol{\sigma}_2$, the grade-2 term, is geometrically interpreted as an oriented, unit plane-segment, and in this case the unit pseudoscalar for the entire space. There are no more unique products to take, and so we have generated the *canonical* basis of $\mathcal{G}_2$. Table 2.1 gives a multiplication table for the basis blades of $\mathcal{G}_2$.

Squaring the pseudoscalar

$$(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)^2 = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 = -\boldsymbol{\sigma}_1^2\boldsymbol{\sigma}_2^2 = -1, \tag{2.10}$$

where the associativity of the geometric product and the fact that orthogonal vectors anticommute has been used. In light of (2.10), we denote the unit-pseudoscalar of $\mathcal{G}_2$ by

$$\mathbf{i} = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2. \tag{2.11}$$

Every bivector in this space is necessarily a scalar multiple of the unit pseudoscalar

$$\mathbf{B} = \beta\mathbf{i}, \tag{2.12}$$

where β is a real number, and so the square of every bivector in $\mathcal{G}_2$ is negative

$$\mathbf{B}^2 = (b\mathbf{i})^2 = -b^2, \tag{2.13}$$

where b is a real number. In order to calculate the magnitude of $\mathbf{B}$ we use

$$|\mathbf{B}| = \sqrt{\mathbf{B}\mathbf{B}^{\dagger}}, \tag{2.14}$$

where we have introduced the dagger notation † for the operation of taking the *reverse*. It has the effect of reversing the order of product of multivectors. For a product of vectors it has the effect or reversing the order of the product

$$(\mathbf{ab}...\mathbf{c})^{\dagger} = \mathbf{c}^{\dagger}...\mathbf{b}^{\dagger}\mathbf{a}^{\dagger} = \mathbf{c}...\mathbf{ba}, \tag{2.15}$$

where the last equality holds because the reverse of any vector is equal to itself. In relation to (2.14) the reverse of a bivector $\mathbf{B}$

$$\mathbf{B}^{\dagger} = (b\mathbf{i})^{\dagger} = b\mathbf{i}^{\dagger}, \tag{2.16}$$

where, like vectors, scalars also reverse to themselves. The pseudoscalar however,

$$\mathbf{i}^{\dagger} = (\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)^{\dagger} = \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1 = -\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 = -\mathbf{i} \tag{2.17}$$

picks up an overall negative sign. Thus, $\mathbf{i}^{\dagger}$ is the inverse of $\mathbf{i}$. Since,

$$\mathbf{i}\mathbf{i}^{\dagger} = (\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)^{\dagger} = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1 = 1 \tag{2.18}$$

This is how we were able to compute the magnitude of $\mathbf{B}$,

$$|\mathbf{B}| = \sqrt{\mathbf{B}\mathbf{B}^{\dagger}} = \sqrt{(b\mathbf{i})(b\mathbf{i}^{\dagger})} = \sqrt{b^2\mathbf{i}\mathbf{i}^{\dagger}} = b \tag{2.19}$$

where b is a positive real number. The geometric product of two vectors $\mathbf{a}$ and $\mathbf{b}$ in terms of their coordinate vectors has the form

$$\mathbf{ab} = (a_1\boldsymbol{\sigma}_1 + a_2\boldsymbol{\sigma}_2)(b_1\boldsymbol{\sigma}_1 + b_1\boldsymbol{\sigma}_2) \tag{2.20}$$

$$= a_1b_1 + a_2b_2 + (a_1b_2 - a_2b_1)\mathbf{i}. \tag{2.21}$$

The scalar portion as

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta = a_1 b_1 + a_2 b_2, \tag{2.22}$$

with θ the angle between $\mathbf{a}$ and $\mathbf{b}$. The bivector portion of (2.21) is

$$\mathbf{a} \wedge \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta \, \mathbf{i} = (a_1 b_2 - a_2 b_1)\mathbf{i}, \tag{2.23}$$

and is interpreted as an oriented plane segment, whose magnitude is equal to the area spanned by the parallelogram having sides $\mathbf{a}$ and $\mathbf{b}$. The orientation is given by rotating $\mathbf{a}$ onto $\mathbf{b}$. Since $\mathbf{a} \wedge \mathbf{b}$ is a bivector, its magnitude can be calculated using (2.14). Which, in this case gives

$$|\mathbf{a} \wedge \mathbf{b}| = \sqrt{|\mathbf{a}|^2 |\mathbf{b}|^2 \sin^2(\theta) \mathbf{i}\mathbf{i}^\dagger}. \tag{2.24}$$

The magnitude is simply equal to the unoriented area of of the parallelogram

$$|\mathbf{a} \wedge \mathbf{b}| = \sqrt{|\mathbf{a}|^2 |\mathbf{b}|^2 \sin^2(\theta)} = |a_1 b_2 - a_2 b_1|. \tag{2.25}$$

Note the loss of the pseudoscalar between (2.23) and (2.25).

2.1.1 Left and Right Action

Here we note the action of the pseudoscalar on each of the basis vectors. of the basis. Left multiplication gives

$$\mathbf{i}\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 \boldsymbol{\sigma}_1 = -\boldsymbol{\sigma}_2 \tag{2.26}$$

$$\mathbf{i}\boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 \boldsymbol{\sigma}_2 = \ \boldsymbol{\sigma}_1, \tag{2.27}$$

rotating each basis vector by $\frac{\pi}{2}$ in the clockwise direction. Likewise, right multiplication by $\mathbf{i}$ generates a positive rotation. The effect of right multiplying each basis vector by $\mathbf{i}$ generates a rotation by $\frac{\pi}{2}$ in the positive sense. It is also interesting to note that we have multiplied a grade-1 multivector, by a grade-2 multivector, and as a result the geometric product has returned another vector, preserving the grade of the vector. This is not the case when we multiply our other basis element, the scalar basis element 1. Since the scalars, grade-0 multivectors, commute with all elements of the algebra, left or right multiplication by $\mathbf{i}$ results in

$$\mathbf{i}(1) = \mathbf{i}, \tag{2.28}$$

taking a grade-0 multivector to a grade-2, not preserving grade. The parenthesis of course are not necessary but have been added for visibility. This little asymmetry will be important when, in the next section, we discuss a connection between vectors and complex numbers.

Another important property of the pseudoscalar can be ascertained from (2.26). If we choose not to contract, but instead write

$$\mathbf{i}\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_1(\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1) = \boldsymbol{\sigma}_1\mathbf{i}^\dagger = -\boldsymbol{\sigma}_1\mathbf{i}, \tag{2.29}$$

we see that the pseudoscalar anticommutes with basis vector $\boldsymbol{\sigma}_1$. The same can also be shown for $\boldsymbol{\sigma}_2$, and so anticommutes with all vectors in the $\mathcal{G}_2$.

We end this section by giving a multiplication table for $\mathcal{G}_2$'s basis blades. As well as a table summarizing the different notations introduced in this chapter.

$\mathcal{G}_2$	1	$\boldsymbol{\sigma}_1$	$\boldsymbol{\sigma}_2$	$\mathbf{i}$
1	1	$\boldsymbol{\sigma}_1$	$\boldsymbol{\sigma}_2$	$\mathbf{i}$
$\boldsymbol{\sigma}_1$	$\boldsymbol{\sigma}_1$	1	$\mathbf{i}$	$\boldsymbol{\sigma}_2$
$\boldsymbol{\sigma}_2$	$\boldsymbol{\sigma}_2$	$-\mathbf{i}$	1	$-\boldsymbol{\sigma}_1$
$\mathbf{i}$	$\mathbf{i}$	-$\boldsymbol{\sigma}_2$	$\boldsymbol{\sigma}_1$	-1

Table 2.1: *Multiplication table for $\mathcal{G}_2$ basis blades.*

Notation	Description
a, b	Scalars (real numbers)
$\mathbf{v}$	Vector
$\mathbf{B}$	Bivector
$\mathbf{i}$	Unit Pseudoscalar
$\mathbf{i}b$	Pseudoscalars
A	Multivector
$Z = a + \mathbf{i}b$	Complex Scalar
$A = \mathbf{v} + \mathrm{Z}$	Vector + Complex Scalar
$A = a + v_1\boldsymbol{\sigma}_1 + v_2\boldsymbol{\sigma}_2$ $+b\boldsymbol{\sigma}_1 \wedge \boldsymbol{\sigma}_2$	Basis Form Multivector

Table 2.2: *Notations for $\mathcal{G}_2$ multivectors*

2.1.2 Complex Algebra

We now look to replicate results of complex algebra by finding the appropiate ways to express them using the operations of geometric algebra.

Consider the space of all scalars and denote them by x. This is just the real number line. Now consider the space of all bivectors, which from (2.12) we know to be all multiples of $\mathbf{i}$, denote them by $y\mathbf{i}$. Their sum, which we denote Z,

$$Z = x + \mathbf{i}y, \tag{2.30}$$

has the form of a complex variable $z = x + iy$, $z \in \mathbb{C}$. If we restrict ourselves from the full algebra of $\mathcal{G}_2$, to only working with elements that can be represented as sums of products, where the products can only contain an even number of vectors, we know this to be the *even subalgebra*, and as previously mentioned, it is *isomorphic* to that of $\mathbb{C}$. Mathematically, this is written as

$$\mathcal{G}_2^+ \sim \mathbb{C}, \tag{2.31}$$

where the tilde denotes the isomorphism, meaning that all the elements of one set may be put into a one-to-one correspondence with the other. It should also be mentioned that there is no such “odd subalgebra”, as the odd elements do not form an algebra. To see that Z belongs in the even subalgebra, first consider the position vector $\mathbf{r}$ in two dimensions

$$\mathbf{r} = x\boldsymbol{\sigma}_1 + y\boldsymbol{\sigma}_2. \tag{2.32}$$

If we left multiply $\mathbf{r}$ by $\boldsymbol{\sigma}_1$

$$\boldsymbol{\sigma}_1\mathbf{r} = \boldsymbol{\sigma}_1(x\boldsymbol{\sigma}_1 + y\boldsymbol{\sigma}_2) = x + \mathbf{i}y, \tag{2.33}$$

we have that

$$Z = \boldsymbol{\sigma}_1\mathbf{r}, \tag{2.34}$$

showing Z as an even product of vectors. We can express Z as an even product of vectors, and we can also express the position vector $\mathbf{r}$ as a geometric product of

$$\boldsymbol{\sigma}_1 Z = \mathbf{r}. \tag{2.35}$$

This provides an easy way to visualize spinors in two dimensions by associating them with their complementary vector.

The analogue of complex conjugation, typically denoted z^* or $\overline{z}$, in geometric algebra is reversion. Thus, we have the correspondence

$$z^* = (x + iy)^* = x - iy \quad \leftrightarrow \quad Z^\dagger = (x + \mathbf{i})^\dagger = x - \mathbf{i}y. \tag{2.36}$$

The modulus of a complex number $|z|$ is given by

$$|z| = \sqrt{zz^*} = \sqrt{x^2 + y^2}, \tag{2.37}$$

which means, we can similarly calculate the magnitude of our even multivector Z in essentially the same way we calculated the magnitude of a bivector in (2.19), which is

$$|Z| = \sqrt{ZZ^\dagger} = \sqrt{x^2 + y^2}. \tag{2.38}$$

One of the strengths of complex algebra is the different representations of complex numbers it makes possible. One can easily switch between the Cartesian and polar forms of a complex number when the need arises. Geometric algebra is no different. In fact one might argue that it is even more enlightening, given the relation of vectors to spinors in geometric algebra. Consider, the two dimensional position vector again

$$\mathbf{r} = x\boldsymbol{\sigma}_1 + y\boldsymbol{\sigma}_2. \tag{2.39}$$

Factor $\boldsymbol{\sigma}_1$ to the left

$$\mathbf{r} = \boldsymbol{\sigma}_1(x + \mathbf{i}y). \tag{2.40}$$

We can replace the scalars x and y by their polar equivalents. Letting ϕ be the angle between $\mathbf{r}$ and the $\boldsymbol{\sigma}_1$-axis, we can write (2.40) as

$$\mathbf{r} = \boldsymbol{\sigma}_1(r\cos\phi + \mathbf{i}r\sin\phi), \tag{2.41}$$

where r is the magnitude of the position vector or the associated spinor. That is

$$r = \sqrt{\mathbf{r}^2} = \sqrt{x^2 + y^2} = \sqrt{ZZ^\dagger} = |Z|. \tag{2.42}$$

To put (2.39) fully in polar form, we factor out the scalar r and identify $\cos\phi + \mathbf{i}\sin\phi$ as a "complex" exponential and write

$$\mathbf{r} = \boldsymbol{\sigma}_1(r\mathrm{e}^{\mathbf{i}\phi}). \tag{2.43}$$

When written in this way, $r\mathrm{e}^{\mathbf{i}\phi}$ may be viewed as an operator acting upon $\boldsymbol{\sigma}_1$, with the operator carrying instructions to rotate and dilate $\boldsymbol{\sigma}_1$ into the position vector $\mathbf{r}$ [10]. It should also be mentioned that technically $\mathrm{e}^{\mathbf{i}\phi}$ is a mixed grade multivector since it contains scalar and bivector terms, and is defined by the usual power series

$$\mathrm{e}^{\mathbf{i}\phi} = \sum_{\mathrm{n}=0}^{\infty} \frac{(\mathbf{i}\phi)^{\mathrm{n}}}{\mathrm{n}!} \tag{2.44}$$

To close out this section on complex numbers, we give a taste of how geometric algebra can easily absorb results from complex analysis. Consider the multivector function $W(x,y) = u(x,y) + \mathbf{i}\,v(x,y)$, where $u(x,y)$ and $v(x,y)$ are scalar functions of the real variables x and y. The geometric product of the two dimensional del-operator $\boldsymbol{\nabla}$ with W results in

$$\boldsymbol{\nabla} W = \left(\frac{\partial}{\partial x}\boldsymbol{\sigma}_1 + \frac{\partial}{\partial y}\boldsymbol{\sigma}_2\right)\Big(u(x,y) + \mathbf{i}\,v(x,y)\Big). \tag{2.45}$$

Simple geometric multiplication gives

$$\boldsymbol{\nabla} W = \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\boldsymbol{\sigma}_1 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\boldsymbol{\sigma}_2. \tag{2.46}$$

If we then require $\boldsymbol{\nabla} W = 0$, since $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ (2.46) are linearly independent of one another, the equations

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \tag{2.47}$$

must be satisfied independently, and we obtain the well known Cauchy-Riemann equations.

In the next section we construct $\mathcal{G}_3$ and discuss the nested algebraic structures embedded within.

2.2 Geometric Algebra in Three Dimensions

A general multivector M in $\mathcal{G}_3$ contains grades from zero to three

$$M = \sum_{k=0}^{3} \langle M \rangle_k = \langle M \rangle_0 + \langle M \rangle_1 + \langle M \rangle_2 + \langle M \rangle_3. \tag{2.48}$$

When working with a basis, a general multivector has the form

$$M = \alpha + a_1\boldsymbol{\sigma}_1 + a_2\boldsymbol{\sigma}_2 + a_3\boldsymbol{\sigma}_3 + B_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 + B_2\boldsymbol{\sigma}_3\boldsymbol{\sigma}_1 + B_3\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 + \beta\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3, \tag{2.49}$$

where α, a_i, B_i, and β are real scalars. This can also be written in the basis free form

$$\begin{aligned} M &= \quad s \quad + \quad \mathbf{v} \quad + \quad \mathbf{B} \quad + \quad \mathbf{T} \\ M &= \text{“\textit{scalar}”} + \text{“\textit{vector}”} + \text{“\textit{bivector}”} + \text{“\textit{trivector}”} \end{aligned} \tag{2.50}$$

Although, in practice it is usually more convenient to work with the form

$$M = \alpha + \mathbf{a} + \mathbf{Ib} + \beta\mathbf{I} = \alpha + \mathbf{a} + \mathbf{I}(\beta + \mathbf{b}). \tag{2.51}$$

Here α and β are real scalars, $\mathbf{a}$ and $\mathbf{b}$ are vectors, and $\mathbf{I}$ is the unit pseudoscalar for the space, a trivector. The term $\mathbf{Ib}$ is the geometric product of the vector $\mathbf{b}$ and the unit pseudoscalar, resulting in a bivector. Figure (2.1) gives a visual representation for the geometric primitives that make up $\mathcal{G}_3$.

$$\mathbf{B} = \mathbf{Ib}. \tag{2.52}$$

By representing geometric primitives with basis blades, we are able to carry our geometric intuition into higher dimensions, as the concept generalizes to any number of dimensions.

2.2.1 Multivector Basis

The geometric algebra of three dimensional space is spanned by the set

$$\{1, \boldsymbol{\sigma}_i, \mathbf{I}\boldsymbol{\sigma}_i, \mathbf{I}\} \qquad\qquad \text{with } i = 1, 2, 3. \tag{2.53}$$

A canonical basis can be gernerated for $\mathcal{G}_3$ The geometric algebra of three dimensions, $\mathcal{G}_3$, can be generated the same way as $\mathcal{G}_2$, by taking all possible combinations of the generators. This process, in two dimensions, generated a canonical basis containing four elements. In three dimensions an extra vector generator results in $\mathcal{G}_3$'s canonical basis having a total of $2^3 = 8$ basis blades. In general, a geometric algebra having n-generating vectors has a canonical basis consisting of 2^n basis blades, with the number

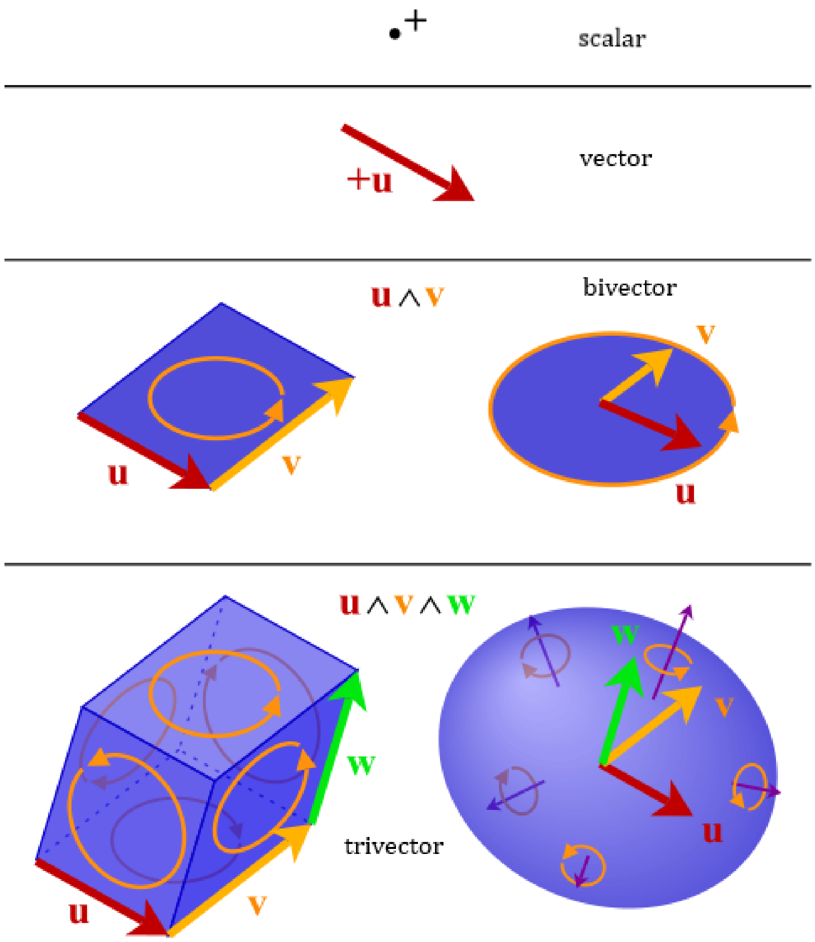

Figure 2.1: *Visualization of $\mathcal{G}_3$ k-blades. Source:k-blades*

of basis blades in any given subspace given by the binomial coefficient. Summing up the basis blades from all sub-spaces gives

$$\sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} = 2^n, \tag{2.54}$$

total basis blades. Table 2.3 gives the multiplication table for $\mathcal{G}_3$'s basis blades.

Since we are dealing with $\mathcal{G}_3$ the pseudoscalars of the space are trivectors, and as a consequence every trivector is necessarily then a scalar multiple of the unit pseudoscalar

$$\mathbf{T} = \beta \mathbf{I} \ . \tag{2.55}$$

1	$\boldsymbol{\sigma}_1$	$\boldsymbol{\sigma}_2$	$\boldsymbol{\sigma}_3$	$\boldsymbol{\sigma}_{12}$	$\boldsymbol{\sigma}_{13}$	$\boldsymbol{\sigma}_{23}$	$\mathbf{I}$
$\boldsymbol{\sigma}_1$	1	$\boldsymbol{\sigma}_{12}$	$\boldsymbol{\sigma}_{13}$	$\boldsymbol{\sigma}_2$	$\boldsymbol{\sigma}_3$	$\mathbf{I}$	$\boldsymbol{\sigma}_{23}$
$\boldsymbol{\sigma}_2$	$-\boldsymbol{\sigma}_{12}$	1	$\boldsymbol{\sigma}_{23}$	$-\boldsymbol{\sigma}_1$	$-\mathbf{I}$	$\boldsymbol{\sigma}_3$	$-\boldsymbol{\sigma}_{13}$
$\boldsymbol{\sigma}_3$	$-\boldsymbol{\sigma}_{13}$	$-\boldsymbol{\sigma}_{23}$	1	$\mathbf{I}$	$-\boldsymbol{\sigma}_1$	$-\boldsymbol{\sigma}_2$	$\boldsymbol{\sigma}_{12}$
$\boldsymbol{\sigma}_{12}$	$-\boldsymbol{\sigma}_2$	$\boldsymbol{\sigma}_1$	$\mathbf{I}$	-1	$-\boldsymbol{\sigma}_{23}$	$\boldsymbol{\sigma}_{13}$	$-\boldsymbol{\sigma}_3$
$\boldsymbol{\sigma}_{13}$	$-\boldsymbol{\sigma}_3$	$-\mathbf{I}$	$\boldsymbol{\sigma}_1$	$\boldsymbol{\sigma}_{23}$	-1	$-\boldsymbol{\sigma}_{12}$	$\boldsymbol{\sigma}_2$
$\boldsymbol{\sigma}_{23}$	$\mathbf{I}$	$-\boldsymbol{\sigma}_3$	$\boldsymbol{\sigma}_2$	$-\boldsymbol{\sigma}_{13}$	$\boldsymbol{\sigma}_{12}$	-1	$-\boldsymbol{\sigma}_1$
$\mathbf{I}$	$\boldsymbol{\sigma}_{23}$	$-\boldsymbol{\sigma}_{13}$	$\boldsymbol{\sigma}_{12}$	$-\boldsymbol{\sigma}_3$	$\boldsymbol{\sigma}_2$	$-\boldsymbol{\sigma}_1$	-1

Table 2.3: *Multiplication table for $\mathcal{G}_3$ basis blades.*

2.2.1.1 The Dual

An important idea in both physics and mathematics is the concept of duality. Within geometric algebra duality has an especially simple manifestation. The *dual* of a multivector A is

$$A^* = A\mathbf{I}^-, \tag{2.56}$$

as in [3] and [11]. Taking the dual of a k-blade $\mathbf{B}_k$, results in an $(n-k)$-blade $\mathbf{B}_k^*$. As a subspace $\mathbf{B}_k^*$ represents the orthogonal complement of $\mathbf{B}_k$ [11]. Care must be taken when dealing with the dual as this definition is not universal, and some prefer the definition to be given by $A^* = A\mathbf{I}$, as in [10] and [6]. Thankfully these definitions differ by at most a sign, and it must be mentioned that many authors, regardless of their preferred definition, simply use the term dual of a multivector to refer to the subspace that represents the orthogonal complement, regardless of the orientation of that subspace. This practice is much more akin to use of duality in differential forms, where the Hodge dual of a k-form is an $n-k$-form, and if you were to take the Hodge dual again, the result is the initial k-form. This slightly differs from duality in geometric algebra. For example based on our definition, the dual of $\boldsymbol{\sigma}_1$ is

$$\boldsymbol{\sigma}_1^* = \boldsymbol{\sigma}_1\mathbf{I}^\dagger. \tag{2.57}$$

If we take the dual again

$$(\boldsymbol{\sigma}_1^*)^* = -\boldsymbol{\sigma}_1, \tag{2.58}$$

we see we pick up an extra negative sign. Geometric algebra, not to be outdone, can actually define the Hodge dual algebraically. For any multivector A the Hodge dual is [3]

$$\star A = A^\dagger \mathbf{I}. \tag{2.59}$$

If take the Hodge dual again

$$\star(\star A) = \star(A^\dagger \mathbf{I}) = (A^\dagger \mathbf{I})^\dagger \mathbf{I}, \tag{2.60}$$

we see that

$$\star(\star A) = \mathbf{I}^\dagger A \mathbf{I} = A, \tag{2.61}$$

where we have made use of the fact that that pseudoscalars commute with all elements of the algebra in three dimensions, and $(A^\dagger)^\dagger$ is returned to itself. This makes the Hodge dual better suited for discussing the relation between two subspaces when one is only interested in the dimensional aspect, rather than the relative orientation between the two subspaces, as is often the case. Nevertheless, geometric algebra's concept of duality is more suitable for programming. This is due to the fact that duality is now a consequence of multiplication with $\mathbf{I}^\dagger$, and does not require any extra rules or special cases to be programmed.

If we take the dual of each $\{\boldsymbol{\sigma}_i\}$:

$$\begin{aligned} \boldsymbol{\sigma}_1^* &= \boldsymbol{\sigma}_1 \mathbf{I}^\dagger = \boldsymbol{\sigma}_3 \boldsymbol{\sigma}_2 \\ \boldsymbol{\sigma}_2^* &= \boldsymbol{\sigma}_2 \mathbf{I}^\dagger = \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_3 \\ \boldsymbol{\sigma}_3^* &= \boldsymbol{\sigma}_3 \mathbf{I}^\dagger = \boldsymbol{\sigma}_2 \boldsymbol{\sigma}_1, \end{aligned} \tag{2.62}$$

we find that a right handed set of vectors are mapped to a set of left handed bivectors. If we now take the products

$$\begin{aligned}
\boldsymbol{\sigma}_1^*\boldsymbol{\sigma}_2^* &= \boldsymbol{\sigma}_3^* \\
\boldsymbol{\sigma}_2^*\boldsymbol{\sigma}_3^* &= \boldsymbol{\sigma}_1^* \\
\boldsymbol{\sigma}_3^*\boldsymbol{\sigma}_1^* &= \boldsymbol{\sigma}_2^*,
\end{aligned} \tag{2.63}$$

we see a very striking resemblance to Hamilton's quaternions. The quaternions, more specifically the unit pure quaternions, $\{i, j, k\}$ satisfy

$$\begin{aligned}
ij &= k \\
jk &= i \\
ki &= j.
\end{aligned} \tag{2.64}$$

They are defined by the equations

$$i^2 = j^2 = k^2 = ijk = -1. \tag{2.65}$$

If we multiply together the three $\{\boldsymbol{\sigma}_k^*\}$ we find

$$\boldsymbol{\sigma}_1^*\boldsymbol{\sigma}_2^*\boldsymbol{\sigma}_3^* = -1, \tag{2.66}$$

which explains a great deal why Hamilton and others struggled to establish the quaternions. They were trying to interpret the quaternions as vectors, which is where the word from, but here we see the real quaternions are a set of left handed bivectors. The product of two pure quaternions $\vec{p}$ and $\vec{q}$ is

$$\begin{aligned}
\vec{p}\vec{q} &= (ai + bj + ck)(xi + yj + zk) \\
\vec{p}\vec{q} &= -\vec{p}\cdot\vec{q} + \vec{p}\times\vec{q},
\end{aligned} \tag{2.67}$$

where $\{a, b, c, x, y, z\}$ are real scalars. This is where Gibbs borrowed from to create his vectorial system, essentially taking everything except the quaternions and their negative inner product. This allowed him to create a conceptually simpler system that was ready for immediate applications. In the next section we show how to cross product relates the two systems.

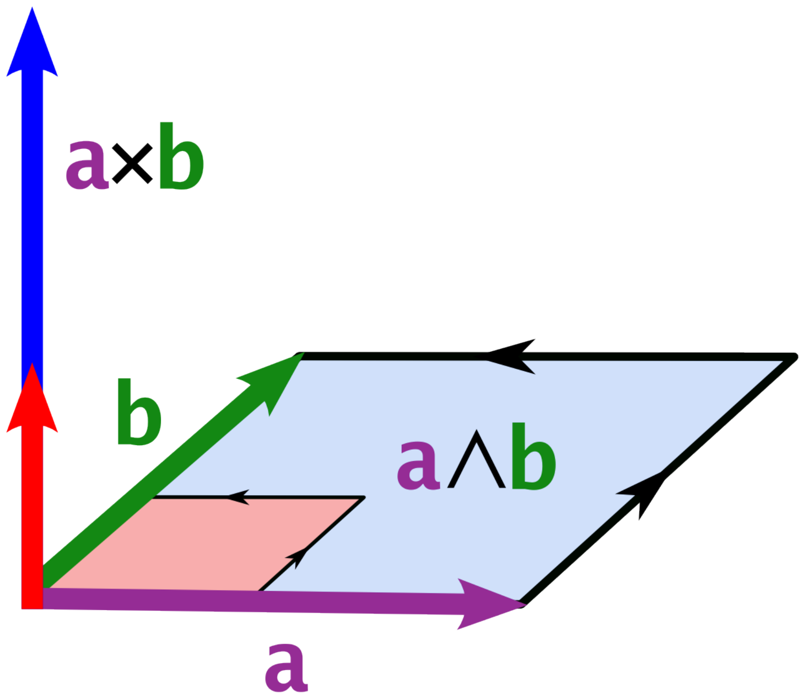

Figure 2.2: *The vector* $\mathbf{a} \times \mathbf{b}$ *is dual to the plane spanned by vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *.*
Source:Cross product

2.2.1.2 Cross Product

The cross product takes as inputs two vectors, **a** and **b**, and produces a third vector **c**, orthogonal to both inputs.

In index notation, with Einstein summation convention in effect

$$\boldsymbol{c} = \boldsymbol{\sigma}_i c_i, \tag{2.68}$$

where the i-th components of **c** is given by

$$c_i = (\mathbf{a} \times \mathbf{b})_i = \varepsilon_{ijk} a_j b_k. \tag{2.69}$$

Inserting (2.69) into (2.68) and summing over i gives

$$\mathbf{c} = \boldsymbol{\sigma}_1 \left(a_2 b_3 - a_3 b_2\right) + \boldsymbol{\sigma}_2 \left(a_3 b_1 - a_1 b_3\right) + \boldsymbol{\sigma}_3 \left(a_1 b_2 - a_2 b_1\right). \tag{2.70}$$

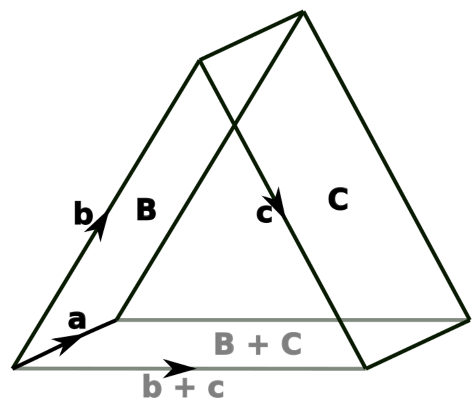

Figure 2.3: *The addition of bivectors* $\mathbf{B}$ *and* $\mathbf{C}$*. Compare with vectors* $\mathbf{b}$ *and* $\mathbf{c}$*.*
Attribution: John Blackburne, CC BY-SA 3.0 Source:Bivector Sum

Compare this with the wedge product of $\mathbf{a}$ and $\mathbf{b}$ in three dimensions

$$\mathbf{a}\wedge\mathbf{b} = \boldsymbol{\sigma}_1\wedge\boldsymbol{\sigma}_2\,(a_1b_2 - a_2b_1) + \boldsymbol{\sigma}_2\wedge\boldsymbol{\sigma}_3\,(a_2b_3 - a_3b_2) + \boldsymbol{\sigma}_3\wedge\boldsymbol{\sigma}_1\,(a_3b_1 - a_1b_3)\,. \quad (2.71)$$

To obtain an equality between the cross product and outer product, we simply take the dual of $\mathbf{a}\wedge\mathbf{b}$, from which we obtain

$$\mathbf{Ic} = \mathbf{Ia}\times\mathbf{b} = \mathbf{a}\wedge\mathbf{b}. \quad (2.72)$$

Figure (2.2) illustrates these relations, graphically.

2.2.2 The Bivector Algebra of $\mathcal{G}_3$

Just as we have three basis vectors in three dimensional space, we have three basis bivectors, and similarly just as we may take a linear combination of basis vectors to form any vector in the space, we may take a linear combination of basis bivectors to form any plane within the space. Bivectors, themselves, satisfy the axioms of a vector space and so can be added and subtracted in the same way, as illustrated in figure (2.3).

We know from equation (2.62) that our basis bivectors have the dual representation $\{\mathbf{I}\boldsymbol{\sigma}_i\}$. The dual representation provides an excellent means for carrying out calculations with bivectors. This is because we can take advantage of the fact that the pseudoscalar commutes with all elements. For instance, if we take the geometric product of the basis bivectors $\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2$ and $\boldsymbol{\sigma}_3\boldsymbol{\sigma}_1$ it is typically easier to write

$$(\mathbf{I}\boldsymbol{\sigma}_3)(\mathbf{I}\boldsymbol{\sigma}_2). \tag{2.73}$$

The geometric product between basis vectors in $\mathcal{G}_3$ can be written in the compact form

$$\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j = \delta_{ij} + \mathbf{I}\boldsymbol{\sigma}_k, \tag{2.74}$$

where $\mathbf{I} = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3$ is the unit pseudo-scalar for the space. It should be noted that (2.74) is only true for unit vectors in $\mathcal{G}_3$, and could equally well be written as

$$\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j = \delta_{ij} + \boldsymbol{\sigma}_i \wedge \boldsymbol{\sigma}_j, \tag{2.75}$$

but the form of (2.74) has been chosen for its resemblance to the formula

$$\hat{\sigma}_i\hat{\sigma}_j = \delta_{ij}\mathbf{1} + i\varepsilon_{ijk}\hat{\sigma}_k, \tag{2.76}$$

where $\mathbf{1}$ is the (2×2) identity matrix and ε_{ijk} is the Levi-Civita symbol. This is of course the well known formula for the product of Pauli matrices. The striking similarity between (2.74) and (2.76) goes far beyond a mere cosmetic resemblance. In fact, the Pauli matrices are nothing more, and nothing less than a matrix representation for the basis vectors of $\mathcal{G}_3$ [8].

CHAPTER 3

SOME TOOLS

3.1 Reflections

Reflections are easily handled within geometric algebra. To reflect a vector $\mathbf{x}$ in some *hyperplane* $\mathbf{B}$, we first represent the hyperplane by its normal vector $\mathbf{n}$. Let us also assume $\mathbf{n}$ is a unit vector, which is a rather reasonable assumption since we can always normalize any vector we are given. Now we decompose $\mathbf{x}$ into parts parallel and perpendicular with respect to $\mathbf{n}$:

$$\mathbf{x} = \mathbf{x}_{\parallel} + \mathbf{x}_{\perp}. \tag{3.1}$$

The portion of $\mathbf{x}$ which is parallel to $\mathbf{n}$, must also be orthogonal to $\mathbf{B}$. The portion of $\mathbf{x}$ perpendicular to $\mathbf{n}$ must lie withing the hyperplane. Thus, we can represent $\mathbf{x}$'s reflection in $\mathbf{B}$ by

$$\mathbf{x}' = \mathbf{x}_{\perp} - \mathbf{x}_{\parallel}. \tag{3.2}$$

Using the associativity of the geometric product, and the fact that $\mathbf{n}$ is a unit vector, we can find explicit expressions for $\mathbf{x}_{\perp}$ and $\mathbf{x}_{\parallel}$:

$$\begin{aligned} \mathbf{x} &= \mathbf{x}\mathbf{n}^2 = (\mathbf{x}\mathbf{n})\mathbf{n} \\ \mathbf{x}_{\parallel} + \mathbf{x}_{\perp} &= (\mathbf{x} \cdot \mathbf{n})\mathbf{n} + (\mathbf{x} \wedge \mathbf{n})\mathbf{n}. \end{aligned} \tag{3.3}$$

The parallel portion of $\mathbf{x}$ is given by

$$\mathbf{x}_{\parallel} = (\mathbf{x} \cdot \mathbf{n})\mathbf{n}, \tag{3.4}$$

which is the projection of $\mathbf{x}$ onto $\mathbf{n}$, which makes $\mathbf{x}_{\perp}$ the rejection of $\mathbf{x}$ by $\mathbf{n}$ and is given by

$$\mathbf{x}_{\perp} = (\mathbf{x} \wedge \mathbf{n})\mathbf{n}. \tag{3.5}$$

If we now insert (3.4) and (3.5) into (3.2)

$$\mathbf{x}' = (\mathbf{x} \wedge \mathbf{n})\mathbf{n} - (\mathbf{x} \cdot \mathbf{n})\mathbf{n}, \tag{3.6}$$

and by noting that the inner product of vectors is commutative while the outer product is anticommutative, we obtain

$$\mathbf{x}' = -(\mathbf{n} \wedge \mathbf{x})\mathbf{n} - (\mathbf{n} \cdot \mathbf{x})\mathbf{n}, \tag{3.7}$$

which we recognize as the geometric product of:

$$\mathbf{x}' = -\mathbf{n}\mathbf{x}\mathbf{n}. \tag{3.8}$$

Nowhere else can such a simple and compact formula be found. Such a compact formulation is not to be undervalued, especially as we compound reflections in the next section to form rotations. It must also be mentioned that (3.8) is valid in any dimension and any signature.

We now make contact with the traditional vector expression for a reflection. Starting with (3.2), we can replace $\mathbf{x}_\perp$ by rearranging (3.1), which allows us to obtain

$$\mathbf{x}' = \mathbf{x} - 2(\mathbf{x} \cdot \mathbf{n})\mathbf{n}, \tag{3.9}$$

the best Gibb's vector algebra has to offer. If we rewrite the dot product in terms of the geometric product

$$\mathbf{x}' = \mathbf{x} - 2\left(\frac{\mathbf{x}\mathbf{n} + \mathbf{n}\mathbf{x}}{2}\right)\mathbf{n} = \mathbf{x} - \mathbf{x}\mathbf{n}^2 - \mathbf{n}\mathbf{x}\mathbf{n} = -\mathbf{n}\mathbf{x}\mathbf{n}, \tag{3.10}$$

we obtain (3.8).

3.1.1 A Simple Example

To give the reader a feel for how computations involving reflections are carried out, we work a simple example with an explicit basis. Suppose we would like to reflect the vector $\mathbf{x} = a\boldsymbol{\sigma_1} + b\boldsymbol{\sigma}_2 + c\boldsymbol{\sigma}_3$ in the xy-plane, where a, b, and c are real constants. The appropriate normal vector to take in this case is $\mathbf{n} = \boldsymbol{\sigma}_3$. Plugging these into (3.8) gives

$$\mathbf{x}' = -\boldsymbol{\sigma}_3 \left(a\boldsymbol{\sigma}_1 + b\boldsymbol{\sigma}_2 + c\boldsymbol{\sigma}_3\right) \boldsymbol{\sigma}_3. \tag{3.11}$$

This reduces to

$$\mathbf{x}' = a\boldsymbol{\sigma_1} + b\boldsymbol{\sigma_2} - c\boldsymbol{\sigma_3}. \tag{3.12}$$

Notice how the anti-commutativity of our basis vectors seamlessly handles the reflection and has the effect of negating only the part of $\mathbf{x}$ that is orthogonal to the plane. In fact, this simple example is just a specific case of a much broader idea a work. The product of orthogonal vectors anti-commute, while the product of parallel vectors commute.

Reflections are an important subset of orthogonal transformations, and as such we should verify that (3.8) does not destroy the invariance of the scalar product. Following [6], we put the grade projection operator to good use and show that the inner product between two reflected vectors remains unchanged. The inner product between two reflected vectors $\mathbf{a}'$ and $\mathbf{b}'$ is given by

$$\mathbf{a}' \cdot \mathbf{b}' = (-\mathbf{nan}) \cdot (-\mathbf{nbn}). \tag{3.13}$$

Since the above equation is a scalar we may use the scalar grade projection operator as follows

$$\mathbf{a}' \cdot \mathbf{b}' = \langle \mathbf{nannbn} \rangle = \langle \mathbf{nabn} \rangle. \tag{3.14}$$

The second equality follows since $\mathbf{n}$ is a unit vector. Next, we take advantage of the cyclic reordering property of the scalar grade projection operator, and permute the terms such that

$$\mathbf{a}' \cdot \mathbf{b}' = \langle \mathbf{abnn} \rangle = \langle \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b} \rangle, \tag{3.15}$$

and finally, we have

$$\mathbf{a}' \cdot \mathbf{b}' = \mathbf{a} \cdot \mathbf{b}. \tag{3.16}$$

Again, the second equality in (3.15) follows by use of the scalar grade projection operator.

3.2 Rotations

Rotations are implemented with the double sided formula,

$$\mathbf{x}' = R\mathbf{x}R^{\dagger}. \tag{3.17}$$

Here, R is an even multivector called a *rotor*. Rotors themselves are the product of two unit vectors, and can be written as

$$R = \mathbf{m}\mathbf{n} = \mathbf{m} \cdot \mathbf{n} + \mathbf{m} \wedge \mathbf{n}. \tag{3.18}$$

The rotation can be thought of first reflecting $\mathbf{x}$ through $\mathbf{n}$, and then reflecting the result through $\mathbf{m}$. That is rotations are built from double reflections. The double sided construction of the rotation formula allows the formula to be generalized to any number of dimensions.

CHAPTER 4

QUANTUM MECHANICS

In this section we make contact with traditional quantum mechanics. We show how complex column spinors, referred to as Pauli spinors, can instead be represented with multivectors. This allows us to give a new interpretation to for bra and ket-vectors, when dealing with two state systems. We then move onto to solving Schrödinger's equation, for a spin-$\frac{1}{2}$ particle in a magnetic field, making it basis free and opening up new routes to obtaining its solution. We also demonstrate how to obtain particular solutions for different initial input data, allowing us to check solutions against traditional methods.

4.1 Pauli Spinors

A Pauli spinor is an element of the complex linear space of two-component column matrices [1], and [3],

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \mathbb{C}^2 \quad \text{with} \quad \psi_1, \psi_2 \in \mathbb{C}. \tag{4.1}$$

Another way of handling spinors within quantum mechanics, apart from the column vector representation, is that of ideal spinors, or square matrix spinors given by

$$\Psi = \begin{pmatrix} \psi_1 & 0 \\ \psi_2 & 0 \end{pmatrix} \in \mathbb{C}(2) \quad \text{with} \quad \psi_1, \psi_2 \in \mathbb{C}. \tag{4.2}$$

The square matrix spinors have an added benefit of being able to represent vectors and spinors within the same algebra. $\mathbb{C}(2)$ is of course the algebra of (2×2) matrices with entries over the complex field. Ideal spinors have the property

$$\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} \psi_1 & 0 \\ \psi_2 & 0 \end{pmatrix} = \begin{pmatrix} \varphi_1 & 0 \\ \varphi_2 & 0 \end{pmatrix}, \tag{4.3}$$

that is, for any $u \in \mathbb{C}(2)$, the product $u\Psi$, results in another square matrix spinor. These elements form a subspace known as a left ideal of $\mathbb{C}(2)$.

Geometric algebra offers a third option for representing spinors. Using the even subalgebra $\mathcal{G}_3$, we may express them as.

$$\psi = a^0 + a^k \mathbf{I}\boldsymbol{\sigma}_k \tag{4.4}$$

Contact can be made with the complex column vector approach using the map [7]

$$\Psi = \begin{pmatrix} a^0 + ia^3 \\ -a^2 + ia^1 \end{pmatrix} \quad \leftrightarrow \quad \psi = a^0 + a^k \mathbf{I}\boldsymbol{\sigma}_k. \tag{4.5}$$

As explained in [6], the $\leftrightarrow$ denotes a one-to-one map between traditional quantum mechanical objects and their multivector equivalents. To see this, we can write Ψ in Dirac notation as

$$|\psi\rangle = \left(a^0 + ia^3\right) \left|z^+\right\rangle + \left(-a^2 + ia^1\right) \left|z^-\right\rangle , \tag{4.6}$$

where $|z^{\pm}\rangle$ represents the spin-up and down states respectively, we can see that the corresponding multivector expression would require the factorization

$$\begin{aligned} \psi &= a^0 + a^1 \mathbf{I}\boldsymbol{\sigma}_1 + a^2 \mathbf{I}\boldsymbol{\sigma}_2 + a^3 \mathbf{I}\boldsymbol{\sigma}_3 \\ &= a^0 + a^3 \mathbf{I}\boldsymbol{\sigma}_3 + \left(-a^2 + a^1 \mathbf{I}\boldsymbol{\sigma}_3\right)\left(-\mathbf{I}\boldsymbol{\sigma}_2\right) . \end{aligned} \tag{4.7}$$

The last equality of (4.7), when compared with (4.5), shows that the traditional i of quantum mechanics is actually the unit bivector $\mathbf{I}\boldsymbol{\sigma}_3 = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2$, which represents the xy-plane. Comparison with (4.6) shows that our spin states are

$$\left|z^+\right\rangle \longleftrightarrow 1 \quad \text{and} \quad \left|z^-\right\rangle \longleftrightarrow -\mathbf{I}\sigma_2. \tag{4.8}$$

We may now express ψ as

$$|\psi\rangle = c_+|z^+\rangle + c_-|z^-\rangle \qquad \Longleftrightarrow \qquad \psi = c_+ + c_-(-\mathbf{I}\boldsymbol{\sigma}_2). \tag{4.9}$$

The "complex" coefficients c_+ and c_-, within geometric algebra, are of the form $c = a + \mathbf{I}\boldsymbol{\sigma}_3 b$.

The eigenstates of the spin operators $\hat{S}_x$ and $\hat{S}_y$, expressed in the $\hat{S}_z$-basis, are

$$\begin{aligned} \left|x^+\right\rangle &= \frac{1}{\sqrt{2}}\left(\left|z^+\right\rangle + \left|z^-\right\rangle\right), \\ \left|x^-\right\rangle &= \frac{1}{\sqrt{2}}\left(\left|z^+\right\rangle - \left|z^-\right\rangle\right), \\ \left|y^+\right\rangle &= \frac{1}{\sqrt{2}}\left(\left|z^+\right\rangle + i\left|z^-\right\rangle\right), \\ \left|y^-\right\rangle &= \frac{1}{\sqrt{2}}\left(\left|z^+\right\rangle - i\left|z^-\right\rangle\right). \end{aligned} \tag{4.10}$$

Using our results from (4.8), we find the eigenstates of $\hat{S}_x$ and $\hat{S}_y$ to be

$$\begin{aligned} \left|x^+\right\rangle \quad &\Longleftrightarrow \quad X^+ = \frac{1}{\sqrt{2}}\left(1 - \mathbf{I}\boldsymbol{\sigma}_2\right) = \frac{1}{\sqrt{2}}\left(1 + \boldsymbol{\sigma}_{13}\right), \\ \left|x^-\right\rangle \quad &\Longleftrightarrow \quad X^- = \frac{1}{\sqrt{2}}\left(1 + \mathbf{I}\boldsymbol{\sigma}_2\right) = \frac{1}{\sqrt{2}}\left(1 - \boldsymbol{\sigma}_{13}\right), \end{aligned} \tag{4.11}$$

where we have used the short hand notation $\boldsymbol{\sigma}_{jk} = \boldsymbol{\sigma}_j\boldsymbol{\sigma}_k$. It is worth mentioning that geometric algebra also allows us to write the equivalent expressions

$$\begin{aligned} X^{\pm} &= \exp(\pm\tfrac{\pi}{4}\boldsymbol{\sigma}_{13}), \\ Y^{\pm} &= \exp(\pm\tfrac{\pi}{4}\boldsymbol{\sigma}_{23}). \end{aligned} \tag{4.12}$$

These are the rotors that one would use to align a vector initially in the x-direction (or y-direction) with the z-axis. Thus we see by choosing the spin operator $\hat{S}_z$ to be diagonal, we have also chosen the x and y basis kets to be rotors that work to align vectors with our preferred chosen axis.

4.2 Spinor Differential Equation

In this section we demonstrate some of the more powerful benefits geometric algebra has to offer Schrödinger's equation

$$i\hbar\frac{d}{dt}|\psi\rangle = \hat{H}|\psi\rangle, \tag{4.13}$$

for a spin-$\frac{1}{2}$ particle in a magnetic field, becomes

$$i\hbar\frac{d}{dt}|\psi\rangle = -\hat{\boldsymbol{\mu}}\cdot\mathbf{B}|\psi\rangle, \tag{4.14}$$

if we ignore spatial dynamics. Here, $\mathbf{B}$ is the magnetic field vector, going against our convention of reserving bold uppercase letters for grade-2 multivectors and higher, but for the good reason of being inline with with long established electromagnetic notational conventions. The magnetic moment operator $\hat{\boldsymbol{\mu}}$ is related to spin operator $\hat{\mathbf{s}}$ by

$$\hat{\boldsymbol{\mu}} = \gamma\hat{\mathbf{s}}, \tag{4.15}$$

where γ is the gyromagnetic ratio, and is given by

$$\gamma = g\frac{q}{2m}. \tag{4.16}$$

Here, g is the experimentally determined spin g-factor, q is the charge of the particle, and its mass is given by m. Using (4.15) and some basic algebra, we can rewrite (4.14) as

$$\frac{d}{dt}|\psi\rangle = \frac{i\gamma}{\hbar}\hat{\mathbf{s}}\cdot\mathbf{B}\;|\psi\rangle \tag{4.17}$$

Since, $\hat{\mathbf{s}}$ is just $\frac{\hbar}{2}$ times the Pauli spin matrices, we can write

$$\frac{d}{dt}|\psi\rangle = \frac{i\gamma}{2}\hat{\sigma}_k B_k|\psi\rangle, \tag{4.18}$$

using the Einstein summation notation. In order to switch from $|\psi\rangle$ to multivector ψ, the action of Pauli matrices, as well as the complex scalar i, upon $|\psi\rangle$ must be exchanged with its corresponding multivector expression. This correspondence can be found in [7], and is given by

$$\hat{\sigma}_k \mid \psi\rangle \quad \longleftrightarrow \quad \boldsymbol{\sigma}_k \psi \boldsymbol{\sigma}_3. \tag{4.19}$$

The appearance of $\boldsymbol{\sigma}_3$ on the right side of ψ, is required to keep ψ within the even subalgebra. There is nothing special about the choice of $\boldsymbol{\sigma}_3$, and is analogous to choosing a basis in which $\hat{\sigma}_3$ is diagonal. To determine the action of $i|\psi\rangle$, first recall from the Pauli matrix algebra

$$\hat{\sigma}_1 \hat{\sigma}_2 \hat{\sigma}_3 = i. \tag{4.20}$$

Substituting (4.20) into (4.19) results in

$$i|\psi\rangle = \hat{\sigma}_1 \hat{\sigma}_2 \hat{\sigma}_3 |\psi\rangle \quad \longleftrightarrow \quad \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 \boldsymbol{\sigma}_3 \psi (\boldsymbol{\sigma}_3)^3 = \mathbf{I} \psi \boldsymbol{\sigma}_3, \tag{4.21}$$

and finally, combining (4.19) and (4.20) gives the final action

$$i\hat{\sigma}_k |\psi\rangle \quad \longleftrightarrow \quad \mathbf{I} \left(\boldsymbol{\sigma}_k \psi \boldsymbol{\sigma}_3 \right) \boldsymbol{\sigma}_3 = \mathbf{I} \boldsymbol{\sigma}_k \psi. \tag{4.22}$$

We can now write the geometric Schrödinger equation for a spin-$\frac{1}{2}$ particle in a magnetic field as

$$\dot{\psi} = \frac{\gamma}{2} \mathbf{I} B_k \boldsymbol{\sigma}_k \psi \ , \tag{4.23}$$

where the over dot denotes a derivative with respect to time. Geometric algebra then shows us that we can write $B_k \boldsymbol{\sigma}_k = \mathbf{B}$, allowing us to put (4.23) in the basis free form

$$\dot{\psi} = \frac{\gamma}{2} \mathbf{I} \mathbf{B} \psi \ . \tag{4.24}$$

This form offers a number of attractive features. One, geometric algebra allows for the decomposition of ψ into the product

$$\psi = \sqrt{\rho} R \ , \tag{4.25}$$

where ρ is a scalar, and R is a rotor satisfying $RR^\dagger = 1$ [7]. The derivative of ψ is then

$$\frac{d}{dt}(\psi) = \left(\frac{\dot{\rho}}{2\sqrt{\rho}}R + \sqrt{\rho}\dot{R}\right) \ . \tag{4.26}$$

Equating (4.26) to (4.24) and replacing ψ on the right hand side by (4.25) gives

$$\dot{\psi} = \left(\frac{\dot{\rho}}{2\sqrt{\rho}}R + \sqrt{\rho}\dot{R}\right) = \frac{\gamma}{2}\mathbf{IB}(\sqrt{\rho}R) \ . \tag{4.27}$$

Note that

$$\psi^\dagger = (\sqrt{\rho}R)^\dagger = R^\dagger(\sqrt{\rho})^\dagger = \sqrt{\rho}R^\dagger \ , \tag{4.28}$$

and since ρ is a scalar, it is unaffected by reversion and commutes with all multivectors. Right multiplying (4.27) by $\psi^\dagger$ leads to

$$\dot{\psi}\psi^\dagger = \left(\frac{\dot{\rho}}{2} + \rho\dot{R}R^\dagger\right) = \frac{\gamma\rho}{2}\mathbf{IB} \ . \tag{4.29}$$

The grade projection operator separates (4.29) into

$$\left\langle \dot{\psi}\psi^\dagger \right\rangle_0 = \frac{\dot{\rho}}{2} = 0 \qquad \text{and} \qquad \left\langle \dot{\psi}\psi^\dagger \right\rangle_2 = \rho\dot{R}R^\dagger = \frac{\gamma\rho}{2}\mathbf{IB} \ . \tag{4.30}$$

The scalar portion, being equal to zero, implies

$$\left\langle \dot{\psi}\psi^\dagger \right\rangle_0 = 0 \qquad \Rightarrow \qquad \rho = \text{a constant} \ . \tag{4.31}$$

The grade-2 term simplifies to

$$\left\langle \dot{\psi}\psi^\dagger \right\rangle_2 = \dot{R}R^\dagger = \frac{\gamma}{2}\mathbf{IB} \ . \tag{4.32}$$

Right multiplying by R gives

$$\dot{R} = \frac{\gamma}{2}\mathbf{IB}R \ . \tag{4.33}$$

Now, if the magnetic field is independent of time, (4.33) can be solved by integration, yielding the solution [6]

$$\psi(t) = \mathrm{e}^{\gamma \mathbf{IB}t/2}\psi(0) \ . \tag{4.34}$$

To determine how states evolve with time from (4.34), we will need to know how the quantum inner product, typically given by

$$\langle \psi \mid \phi \rangle = (\psi_1^* \ \psi_2^*) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} , \tag{4.35}$$

translates into geometric algebra. This is given in [6], and the product translates as

$$\langle \psi \mid \phi \rangle \quad \leftrightarrow \quad \left\langle \psi^\dagger \phi \right\rangle_0 - \left\langle \psi^\dagger \phi \mathbf{I} \boldsymbol{\sigma}_3 \right\rangle_0 \mathbf{I} \boldsymbol{\sigma}_3 \tag{4.36}$$

Care must be taken not to confuse the quantum inner product $\langle \psi \mid \phi \rangle$, and the scalar grade projection operator $\langle \psi^\dagger \phi \rangle_0$, which both make use of angled brackets. Sticking with the notation in [6], the quantum inner product within geometric algebra is denoted

$$\left\langle \psi^\dagger \phi \right\rangle_q = \quad \left\langle \psi^\dagger \phi \right\rangle_0 - \left\langle \psi^\dagger \phi \mathbf{I} \boldsymbol{\sigma}_3 \right\rangle_0 \mathbf{I} \boldsymbol{\sigma}_3. \tag{4.37}$$

We see the product picks out the scalar portion of the geometric product, as well as projecting out the portion in the quantization plane $\mathbf{I}\boldsymbol{\sigma}_3$.

Returning to the problem of a spin-$\frac{1}{2}$ particle in a magnetic field, if the state is initially known to be in the spin state $|\psi(0)\rangle = |x^+\rangle = \frac{1}{\sqrt{2}}\left(|z^+\rangle + |z^-\rangle\right)$ and we take $\mathbf{B} = B_0 \boldsymbol{\sigma}_3$, we can determine the state at some later time t using (4.34). Typically, this is done using the time evolution operator $\hat{U}(t)$ and would be written

$$|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle \ . \tag{4.38}$$

Comparing with (4.34), we see that the rotor $R(t) = \mathrm{e}^{\gamma \mathbf{I} \mathbf{B} t/2}$ plays the role of time evolution operator, and so the state function at some later time t is then

$$\psi(t) = \mathrm{e}^{\boldsymbol{\sigma}_{12}\omega_0 \mathrm{t}/2} \psi(0) = \mathrm{e}^{\boldsymbol{\sigma}_{12}\omega_0 \mathrm{t}/2}\, \mathrm{X}^+ = \mathrm{e}^{\boldsymbol{\sigma}_{12}\omega_0 \mathrm{t}/2} \left(1 - \mathbf{I}\boldsymbol{\sigma}_2\right)/\sqrt{2}. \tag{4.39}$$

where we have written $\gamma B_0 = \omega_0$. As an example, say we are interested in determining the probability of finding the particle in the state $|x^+\rangle$. First we calculate the probability amplitude using (4.37)

$$\left\langle \left(X^{+}\right)^{\dagger} \psi(t) \right\rangle_{q} = \left\langle (X^{+})^{\dagger} \psi(t) \right\rangle_{0} - \mathbf{I}\boldsymbol{\sigma}_3 \left\langle (X^{+})^{\dagger} \psi(t) \mathbf{I}\boldsymbol{\sigma}_3 \right\rangle_{0}. \tag{4.40}$$

To simplify the calculation, we can first write $\psi(t) = R(t)X^{+}$ and then use the cyclic reordering property of the scalar grade projection operator. This simplifies the scalar term of (4.40) to

$$\begin{aligned} \left\langle \left(X^{+}\right)^{\dagger} R(t)X^{+} \right\rangle_{0} &= \left\langle R(t)X^{+} \left(X^{+}\right)^{\dagger} \right\rangle_{0} \\ &= \langle R(t) \rangle_{0} = \left\langle \mathrm{e}^{\boldsymbol{\sigma}_{12}\omega_0 \mathrm{t}/2} \right\rangle_{0} = \cos\left(\omega_0 t/2\right). \end{aligned} \tag{4.41}$$

The $\mathbf{I}\boldsymbol{\sigma}_3$ term of (4.40) contains no scalar portion and so vanishes. The probability $|\langle x^{+} \mid \psi(t)\rangle|^2$ is then obtained by squaring the result of (4.41), giving

$$|\langle x^{+} \mid \psi(t)\rangle|^2 = \cos^2\left(\omega_0 t/2\right), \tag{4.42}$$

as expected.

The expectation value of spin in the k-direction is usually given the expression

$$\langle \mathbf{s}_k \rangle = \langle \psi(t) \left| \hat{\mathbf{s}}_k \right| \psi(t) \rangle . \tag{4.43}$$

This is easily ported over to geometric algebra by first using (4.19) to obtain

$$\hat{\mathbf{s}}_k \mid \psi(t)\rangle \quad \longleftrightarrow \quad \mathbf{s}_k \psi(t) \boldsymbol{\sigma}_3, \tag{4.44}$$

and then using (4.37) as before. This result of which is

$$\langle \psi(t) \left| \hat{\mathbf{s}}_k \right| \psi(t) \rangle \longleftrightarrow \left\langle \psi^{\dagger} \boldsymbol{\sigma}_k \psi \boldsymbol{\sigma}_3 \right\rangle_0 - \mathbf{I}\boldsymbol{\sigma}_3 \left\langle \psi^{\dagger} \boldsymbol{\sigma}_k \psi \boldsymbol{\sigma}_3 \mathbf{I}\boldsymbol{\sigma}_3 \right\rangle_0 . \tag{4.45}$$

Since the pseudoscalar commutes with all elements in three dimensions, this may be simplified to

$$\left\langle \psi^{\dagger} \boldsymbol{\sigma}_k \psi \boldsymbol{\sigma}_3 \right\rangle_0 - \mathbf{I}\boldsymbol{\sigma}_3 \left\langle \psi^{\dagger} \mathbf{I} \boldsymbol{\sigma}_k \psi \right\rangle_0 \tag{4.46}$$

Under reversion, the term $\left\langle \psi^\dagger \mathbf{I} \boldsymbol{\sigma}_k \psi \right\rangle_0$ is equal to the negative of itself, and so cannot be a scalar and therefore vanishes. The remaining term can be cyclically permuted to give

$$\left\langle \boldsymbol{\sigma}_k \psi \boldsymbol{\sigma}_3 \psi^\dagger \right\rangle_0 . \tag{4.47}$$

In [7] the quantity $\psi \boldsymbol{\sigma}_3 \psi^\dagger$ is used to define the spin vector

$$\mathbf{s} = \psi \boldsymbol{\sigma}_3 \psi^\dagger. \tag{4.48}$$

Rewriting (4.47) with this definition gives a compact form for the spin expectation value

$$\left\langle \boldsymbol{\sigma}_k \psi \boldsymbol{\sigma}_3 \psi^\dagger \right\rangle_0 = \langle \boldsymbol{\sigma}_k \, \mathbf{s} \rangle_0 = \boldsymbol{\sigma}_k \cdot \mathbf{s}. \tag{4.49}$$

Aside from being compact, the quantum expectation also carries with it a slightly different interpretation than its traditional counterpart. The traditional expression is thought of as an expectation for a quantum operator, whereas in geometric algebra we are simply projecting out the component of spin vector along the k-direction.

Returning once again to our spin-$\frac{1}{2}$ particle in a magnetic field, we can now calculate the spin expectation value, say for the x-direction. We first form the spin vector using (4.48), with $\psi(t) = R(t)\psi(0)$. Since our initial state is given as X^+, and we determined $R(t) = \exp(\mathbf{I}\boldsymbol{\sigma}_3 \omega_0 t/2)$, our spin vector is then

$$\mathbf{s} = \psi \boldsymbol{\sigma}_3 \psi^\dagger = \left(R X^+\right) \boldsymbol{\sigma}_3 \left(R X^+\right)^\dagger = R \left(X^+ \boldsymbol{\sigma}_3 (X^+)^\dagger\right) R^\dagger. \tag{4.50}$$

We recognize $\left(X^+ \boldsymbol{\sigma}_3 (X^+)^\dagger\right)$ as the transformation taking $\boldsymbol{\sigma}_3 \to \boldsymbol{\sigma}_1$, reducing the spin vector to

$$\mathbf{s} = R \boldsymbol{\sigma}_1 R^\dagger. \tag{4.51}$$

Expanding the above, the spin vector is then

$$\mathbf{s} = \tfrac{\hbar}{2} \cos(\omega_0 t) \boldsymbol{\sigma}_1 - \tfrac{\hbar}{2} \sin(\omega_0 t) \boldsymbol{\sigma}_2. \tag{4.52}$$

The expected value of spin in the x-direction is then

$$\langle s_1 \rangle = \boldsymbol{\sigma}_1 \cdot \mathbf{s} = \frac{\hbar}{2} \cos(\omega_0 t) \,. \tag{4.53}$$

4.3 Variable Magnetic Field

We now move on to another example of a rotor differential equation. The setup is the same as the preceding problem, but now suppose the magnetic field is a function of time and is given by $\mathbf{B}(t) = (B_1 \cos(\omega t), B_1 \sin(\omega t), B_0)$. Taking (4.24) as our starting point, we have

$$\dot{\psi} = \frac{\gamma}{2} \mathbf{I} \mathbf{B} \psi = \frac{\gamma}{2} \mathbf{I} \left(B_1 \boldsymbol{\sigma}_1 \cos(\omega t) + B_1 \boldsymbol{\sigma}_2 \sin(\omega t) + B_0 \boldsymbol{\sigma}_3 \right) \psi. \tag{4.54}$$

Again, geometric algebra offers a unique route to the solution not available to traditional methods. The strategy here is again to lock up the time dependence, and hence the dynamics, into a rotor $R(t)$. We begin by factoring the magnetic field into

$$\begin{aligned} B &= B_1 \boldsymbol{\sigma}_1 \left(\cos(\omega t) + \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 \sin(\omega t) \right) + B_0 \sigma_3 \\ &= B_1 \boldsymbol{\sigma}_1 \exp\left(\mathbf{I} \boldsymbol{\sigma}_3 \omega t \right) + B_0 \boldsymbol{\sigma}_3. \end{aligned} \tag{4.55}$$

Since, $\boldsymbol{\sigma}_1$ lies in the plane represented by the bivector $\boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2$, it anticommutes with $\exp(\mathbf{I} \boldsymbol{\sigma}_3 \omega t)$, which generates rotations in the $\boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2$- plane. This allows us to write $\mathbf{B}$ as

$$B = \exp\left(-\mathbf{I} \boldsymbol{\sigma}_3 \omega t / 2 \right) B_1 \boldsymbol{\sigma}_1 \exp\left(\mathbf{I} \boldsymbol{\sigma}_3 \omega t / 2 \right) + B_0 \boldsymbol{\sigma}_3 \tag{4.56}$$

The fact that $\boldsymbol{\sigma}_3$ is orthogonal to the plane of rotation allows us to write the term $B_0 \boldsymbol{\sigma}_3$ as

$$B_0 \boldsymbol{\sigma}_3 = \exp(-\mathbf{I} \boldsymbol{\sigma}_3 \omega t / 2) B_0 \boldsymbol{\sigma}_3 \exp(\mathbf{I} \boldsymbol{\sigma}_3 \omega t / 2). \tag{4.57}$$

As in [6], we can define

$$S = \exp(-\mathbf{I} \boldsymbol{\sigma}_3 \omega t / 2), \tag{4.58}$$

which allows us to write the magnetic field as

$$\mathbf{B} = S(B_1\boldsymbol{\sigma}_1 + B_0\boldsymbol{\sigma}_3)S^\dagger. \tag{4.59}$$

If we let $\mathbf{B}_c = B_1\boldsymbol{\sigma}_1 + B_0\boldsymbol{\sigma}_3$, then (4.54) takes the simple form

$$\dot{\psi} = \tfrac{\gamma}{2}\mathbf{I}S\,\mathbf{B}_c S^\dagger\psi. \tag{4.60}$$

Again, we have locked up the time-dependence in a rotor. To solve this rotor differential equation, first note that

$$\frac{d}{dt}\left(S^\dagger\psi\right) = \dot{S}^\dagger\psi + S^\dagger\dot{\psi}. \tag{4.61}$$

We can easily form the first term $\dot{S}^\dagger\psi$ after finding

$$\frac{d}{dt}\left(S^\dagger\right) = \tfrac{1}{2}\mathbf{I}\boldsymbol{\sigma}_3\omega S^\dagger. \tag{4.62}$$

The second term $S^\dagger\dot{\psi}$ is found by left multiplying (4.60) by $S^\dagger$, resulting in

$$S^\dagger\dot{\psi} = \tfrac{\gamma}{2}\mathbf{I}\mathbf{B}_c S^\dagger\psi. \tag{4.63}$$

Combining (4.62) and (4.63) we can rewrite (4.61) as

$$\frac{d}{dt}(S^\dagger\psi) = \tfrac{1}{2}\mathbf{I}\left(\boldsymbol{\sigma}_3\omega + \gamma\mathbf{B}_c\right)S^\dagger\psi, \tag{4.64}$$

a first order differential equation. Yet again, geometric algebra has allowed us to find a first order differential equation. In the typical approach, one is forced to work with the complex components of the column spinor $|\psi\rangle$. Doing so, then forces one to work with a pair of coupled, second-order differential equations as a consequence. By working with the first order equation (4.64) we can immediately solve again by inspection. The solution is

$$S^\dagger\psi = \exp\left(\mathbf{I}\left(\omega\boldsymbol{\sigma}_3 + \gamma\mathbf{B}_c\right)t/2\right)\psi(0). \tag{4.65}$$

Solving for ψ, plugging in the explicit expression for $\mathbf{B}_c$, and writing $\omega_1 = \gamma B_1$ and $\omega_0 = \gamma B_0$, we arrive at

$$\psi(t) = \exp\left(-\mathbf{I}\boldsymbol{\sigma}_3 \omega t/2\right) \exp\left(\mathbf{I}\left(\omega_1 \boldsymbol{\sigma}_1 + \left(\omega_0 + \omega\right)\boldsymbol{\sigma}_3\right) t/2\right) \psi(0). \tag{4.66}$$

4.4 Spacetime Algebra

The geometric algebra $\mathcal{G}_{1,3}$ is known as the spacetime algebra (STA). It is generated by the four vectors $\{\gamma_\mu\}$. Similar to Dirac's gamma matrices, they satisfy the equations

$$\gamma_0^2 = 1,\ \gamma_k^2 = -1, \qquad \text{and} \qquad \gamma_\mu \cdot \gamma_\nu = 0 \quad (\mu \neq \nu). \tag{4.67}$$

One of the more interesting aspects of this algebra is its bivector algebra. There are a total of six bivectors. They fall into two classes

$$\left(\gamma_k \wedge \gamma_0\right)^2 = +1 \qquad \text{and} \qquad \left(\gamma_j \wedge \gamma_k\right)^2 = -1, \tag{4.68}$$

those with positive square and those with negative square. The positive squaring bivectors contain the timelike vector γ_0. They are interpreted as representing a relative vector in the time frame defined by γ_0.

An event in spacetime is represented by the four dimensional vector

$$x = x^\mu \gamma_\mu = ct\gamma_0 + x^i \gamma_i. \tag{4.69}$$

When right multiplied by γ_0

$$x\gamma_0 = ct + x^i \gamma_i \gamma_0. \tag{4.70}$$

they become a multivector equivalent to a four-vector. Four vectors are a combination of scalar and three dimensional vector. Multiplying a vector in the spacetime algebra by γ_0 is known as the spacetime split. It splits a homogenous spacetime vector into a sum of a scalar and spacetime bivector. Spacetime bivectors generated through the spacetime split are taken as defining a rest frame of relative vectors. This motivates the following notation

$$x\gamma_0 = ct + x^i\gamma_i\gamma_0 = ct + x^i\boldsymbol{\sigma}_i, \tag{4.71}$$

and so we arrive at the expression

$$x\gamma_0 = ct + \mathbf{x}, \tag{4.72}$$

which contains the same amount of information as a four vector. The spacetime split makes it explicit whether or not an expression is observer dependent. Any expression containing a timelike vector defines an observer for that frame. This makes determining whether or not an expression represents an invariant nearly trivial. Our version of a four vector is obviously observer dependent, but if we are interested in an invariant lenght, independent of any observer, we simply write

$$x^2 = x\gamma_0\gamma_0 x = (ct + \mathbf{x})(ct - \mathbf{x}) = (c^2t^2 - \mathbf{x}^2), \tag{4.73}$$

making contracting four-vectors as easy as multiplying polynomials.

The identification of $\boldsymbol{\sigma}_k = \gamma_k\gamma_0$ mean that we can interpret spacetime bivectors as relative vectors in three dimensional space, so we are aided by being able to maintain meaningful notation. This also allows us to write the 16 dimensional basis for the STA as

$$1, \quad \{\gamma_\mu\}, \quad \{\boldsymbol{\sigma}_k, \mathbf{I}\boldsymbol{\sigma}_k\}, \quad \{\mathbf{I}\gamma_\mu\}, \quad \mathbf{I}. \tag{4.74}$$

The even subalgebra of this basis

$$1, \quad \{\boldsymbol{\sigma}_k\}, \quad \{\mathbf{I}\boldsymbol{\sigma}_k\}, \quad \mathbf{I}, \tag{4.75}$$

will be recognized as a basis for $\mathcal{G}_3$, and so geometric algebra provides incredible tool not only for performing calculations, but is also invaluable for the way it seamlessly transitions between relativistic and non-relativistic arenas.

The Dirac equation for a free particle is given by the equation

$$(i\hbar\gamma^{\mu}\partial_{\mu} - mc)\,\psi = 0, \tag{4.76}$$

which in natural units ($\hbar$ = c = 1) has the simplified form

$$i\gamma^{\mu}\partial_{\mu}\psi = m\psi. \tag{4.77}$$

The STA version of the Dirac equation is written

$$\nabla\psi\mathbf{I}\boldsymbol{\sigma}_3 = m\psi\gamma_0 \tag{4.78}$$

The Dirac operator ∇, when viewed through the lens of geometric algebra, is the simple the vector derivative. We can take the ∇ and expand it relative to the frame vectors $\{\gamma_{\mu}\}$, along with appropriate coordinates as

$$\nabla = \gamma^{\mu}\frac{\partial}{\partial x^{\mu}} = \gamma_0\frac{\partial}{\partial t} + \gamma^{i}\frac{\partial}{\partial x^{i}}. \tag{4.79}$$

It may be noticed that γ_{μ} vectors now appear with an upper index. The upper index indicates that these vectors belong to the reciprocal frame. These vectors are necessary as $\{\gamma_1, \gamma_2, \gamma_3\}$ have negative square. Taking γ_1 as an example, to find its reciprocal vector we can simply solve

$$\gamma_1\gamma^1 = 1, \tag{4.80}$$

left multiplication gives

$$\gamma^1 = -\gamma_1. \tag{4.81}$$

This ensures the proper spacetime split

$$\nabla\gamma_0 = \partial_t + \gamma^{i}\gamma_0\partial_i = \partial_t - \boldsymbol{\nabla}. \tag{4.82}$$

CHAPTER 5

CONCLUSIONS AND FUTURE WORK

We have shown that the robust approach to Clifford algebra, geometric algebra, inspires new ways of visualizing, interpreting, and understanding quantum mechanics. We traced how geometric algebra leads one to consider a new definition of spinor, one in which it is represented by the multivectors of $\mathcal{G}_3^+$, and can be visualized as planes of rotation. The use of basis free representations of multivectors was seen to open up new avenues, providing routes unavailable to other formalisms, to solve quantum mechanical problems. We also uncovered a new way of understanding and visualizing Dirac's bra-ket formalism. Geometric algebra revealed these to be rotors, the rotors necessary to align a spin vector perpendicular to a quantization plane, determined by some measuring apparatus.

5.1 Future Work

Geometric algebra proved to be an excellent tool for understanding the quantum two-state system. An important two-state system is that of the ammonia molecule NH_3. One might think that studying the ammonia molecule should be no different than the two state systems presented in this paper. However, the Hamiltonian associated with this problem, is considerably different than those presented here. This is due to the fact that once it represented in geometric algebra, the Hamiltonian is actually a scalar plus a vector. The Hamiltonians presented here turned out to be purely vectors, which made obtaining solutions easier within geometric algebra than their traditional counterparts. Recall, in the case of the oscillating magnetic field, the tools and techniques of geometric algebra made it possible to obtain a linear, first order differential equation, which is not possible using traditional techniques, where one works with the components of the spinor separately, resulting in a pair of coupled, second order differential equations. Aside from being easier to solve, first order equations are also numerically more stable than second order, a very attractive feature. The ammonia molecule does not yield a solution as easily as one might expect. This is due to an extra scalar term in the Hamiltonian, which is seen to arise from the symmetry of the molecule. Due to the presence of this term, the associated differential equation is somewhat more complicated. However, it may be possible to "rotate" these terms away, such that they vanish all together, making it possible to obtain a first order equation for the ammonia molecule as well.

The algebras presented in this work, all have one thing in common. They are all nested within $\mathcal{G}(1,3)$, known as the spacetime algebra. We saw that the algebra of the complex numbers, could be identified with the even subalgebra of $\mathcal{G}_2$. The extension of the complex numbers, the quaternions, are the even subalgebra of G_3, the algebra of three dimensional space. The algebra of space is nested within the algebra of Minkowski spacetime. What important structure might $\mathcal{G}(1,3)$ be nested in, and what implications might be determined from identifying this structure. Could this suggest a route to Grand Unification, or at the very least, rule some paths out?

GLOSSARY

grade The *grade* of a multivector is a nonnegative integer ' '. 5

grade-projection operator The *grade-projection operator* has the following properties

$$\langle A + B\rangle_k = \langle A\rangle_k + \langle B\rangle_k$$

$$\langle \lambda A\rangle_k = \lambda\langle A\rangle_k = \langle A\rangle_k\lambda$$

$$\langle\langle A\rangle_k\rangle_k = \langle A\rangle_k,$$

where A and B are general multivecotrs, and λ is a real number. 6

hyperplane A *hyperplane* is the subspace whose dimension is one less than that of its ambient space.These subspaces are represented by $(n-1)$-blades $\mathbf{B}_{n-1}$. 31

k-blade A *k-blade* $(k > 0)$ $\mathbf{A}_k$ in $\mathcal{G}_n$ is a geometric product of k distinct, orthogonal vectors

$$\mathbf{A}_k = \mathbf{a}_1 ... \mathbf{a}_k.$$

A 0-blade is a nonzero scalar. 5

k-vector A *k-vector* is a linear combination of k-blades. 5

multivector *Multivectors* are elements of a geometric algebra. 5

reverse For a k-blade $\mathbf{A}_k$ the *reverse* is defined by

$$\mathbf{A}_k^\dagger = \mathbf{a}_k\mathbf{a}_{k-1}...\mathbf{a}_2\mathbf{a}_1$$

The formula below allows us to quickly calculate the reverse of $\mathbf{A}_k$

$$\mathbf{A}_k^\dagger = (-1)^{k(k-1)/2}\mathbf{A}_k.$$

For a general multivector A apply $\dagger$ to each k-vector term. 16

BIBLIOGRAPHY

[1] Rafal Ablamowicz, Garret Sobczyk, et al. *Lectures on Clifford (geometric) algebras and applications*. Springer, 2004.

[2] G Aragon, JL Aragón, and MA Rodríguez. Clifford algebras and geometric algebra. *Advances in Applied Clifford Algebras*, 7(2):91–102, 1997.

[3] Eduardo Bayro-Corrochano. *Geometric algebra applications vol. I: Computer vision, graphics and neurocomputing*. Springer, 2018.

[4] William Clifford. Applications of grassmann's extensive algebra. *American Journal of Mathematics*, 1(4):350–358, 1878.

[5] Michael J Crowe. *A history of vector analysis: The evolution of the idea of a vectorial system*. Courier Corporation, 1994.

[6] Chris Doran and Anthony Lasenby. *Geometric Algebra for Physicists*. Cambridge University Press, 2003.

[7] Chris Doran, Anthony Lasenby, and Stephen Gull. States and operators in the spacetime algebra. *Foundations of physics*, 23(9):1239–1264, 1993.

[8] Justin Dressel, Konstantin Y Bliokh, and Franco Nori. Spacetime algebra as a powerful tool for electromagnetism. *Physics Reports*, 589:1–71, 2015.

[9] Desmond Fearnley-Sander. Hermann grassmann and the creation of linear algebra. *The American Mathematical Monthly*, 86(10):809–817, 1979.

[10] David Hestenes. *New foundations for classical mechanics*, volume 99 of *Fundamental theories of physics*. Kluwer, 1999.

[11] Alan Macdonald. A survey of geometric algebra and geometric calculus. *Advances in Applied Clifford Algebras*, 27(1):853–891, 2017.

[12] John Archibald Wheeler. *Geometrodynamics*, volume 1. Academic Press, 1962.