Suppose that $F$ is a partially ordered field with a directed partial order and $K$ is a non-archemedean totally ordered subfield of $F$ with $\mathrm{K}_{+}=\mathrm{F}+\cap \mathrm{K} K+=F+\cap K$. In this note, directed partial orders are constructed for complex numbers and quaternions over $F$. It is also shown that real quaternions cannot be made into a directed algebra over the real field with the total order.

Let $T$ be a non-archimedean totally ordered field and $\mathrm{CT}=\mathrm{T}+\mathrm{Ti} C T=T+T i$ be the field of complex numbers over $T$, where $\mathrm{i}_{2}=-1 i 2=-1$, and $\mathrm{HT}=\mathrm{T}+\mathrm{Ti}+\mathrm{Tj}+\mathrm{Tk} H T=T+T i+T j+T k$ be the division algebra of quaternions over T. In [4], a general method of constructing directed partial orders on Ст $С$ T and Нт $H T$ has been developed. The purpose of this note is to generalize the results in [4] to complex numbers and quaternions over non-archimedean partially ordered fields with a directed partial order.

We review a few definitions and the reader is referred to [2, 4] for undefined terminology and background on partially ordered rings and lattice-ordered rings ( $\ell \ell$-rings). For a partially ordered ring $R$, the positive cone is defined as $\mathrm{R}_{+}=\{\mathrm{r} \in \mathrm{R} \mid \mathrm{r} \geq 0\} R+=\{r \in R / r \geq 0\}$. The partial order on a partially ordered $\operatorname{ring} R$ is called directed if each element of $R$ can be written as a difference of two positive elements of $R$. A partially ordered ring (algebra) with a directed partial order is called a directed ring (algebra). In the following, we always assume that $F$ is a partially ordered field with a directed partial order and $K$ is a non-archimedean totally ordered subfield of $F$, so $\mathrm{F}+\cap \mathrm{K}=\mathrm{K}+F+\cap K=K+$.

Following result will be used in the proof of main results in the paper.

## Lemma 1

Suppose that $R$ is a partially ordered ring with the property that for any $\mathrm{r} \in \mathrm{R} r \in R, 2 \mathrm{r} \geq 02 r \geq 0$ implies that $\mathrm{r} \geq 0 r \geq 0$. Then for any $\mathrm{x}, \mathrm{y} \in \mathrm{R}+x, y \in R+$, and $\mathrm{a}, \mathrm{b} \in \mathrm{Ra}, b \in R,-\mathrm{x} \leq \mathrm{a} \leq \mathrm{x}-x \leq a \leq x$ and $-\mathrm{y} \leq \mathrm{b} \leq \mathrm{y}-\mathrm{y} \leq b \leq y$ implies that $-x y \leq a b \leq x y-x y \leq a b \leq x y$.

Proof
From $-\mathrm{x} \leq \mathrm{a} \leq \mathrm{x}-x \leq a \leq x$ and $-\mathrm{y} \leq \mathrm{b} \leq \mathrm{y}-y \leq b \leq y$, we have

$$
\begin{equation*}
(\mathrm{x}+\mathrm{a})(\mathrm{y}-\mathrm{b}) \geq 0 \Rightarrow \mathrm{xy}+\mathrm{ay}-\mathrm{xb}-\mathrm{ab} \geq 0,(x+a)(y-b) \geq 0 \Rightarrow x y+a y-x b-a b \geq 0, \tag{1}
\end{equation*}
$$

and
$(\mathrm{x}-\mathrm{a})(\mathrm{y}+\mathrm{b}) \geq 0 \Rightarrow \mathrm{xy}-\mathrm{ay}+\mathrm{xb}-\mathrm{ab} \geq 0 .(x-a)(y+b) \geq 0 \Rightarrow x y-a y+x b-a b \geq 0$.

Adding (1) and (2), we have $2(x y-a b) \geq 02(x y-a b) \geq 0$, so $x y-a b \geq 0 x y-a b \geq 0$ and $x y \geq a b x y \geq a b$.

From $-\mathrm{x} \leq \mathrm{a} \leq \mathrm{x}-x \leq a \leq x$ and $-\mathrm{y} \leq \mathrm{b} \leq \mathrm{y}-y \leq b \leq y$, we also have
$(x+a)(y+b) \geq 0 \Rightarrow x y+a y+x b+a b \geq 0,(x+a)(y+b) \geq 0 \Rightarrow x y+a y+x b+a b \geq 0$,
and
$(x-a)(y-b) \geq 0 \Rightarrow x y-a y-x b+a b \geq 0 .(x-a)(y-b) \geq 0 \Rightarrow x y-a y-x b+a b \geq 0$.

Adding (3) and (4), we have $2(x y+a b) \geq 02(x y+a b) \geq 0$, so $x y+a b \geq 0 x y+a b \geq 0$, that is, $\mathrm{ab} \geq-x y a b \geq-x y$.

We notice that since $F$ contains a totally ordered subfield $K$, for any $\mathrm{a} \in \mathrm{Fa} \in F$, $2 \mathrm{a} \geq 02 \mathrm{a} \geq 0$ implies that $\mathrm{a} \geq 0 a \geq 0$. In fact, since $0<2 \in \mathrm{~K} 0<2 \in K$, $0<12 \in \mathrm{~K} 0<12 \in K$, so $\mathrm{a}=12(2 \mathrm{a}) \geq 0 a=12(2 \mathrm{a}) \geq 0$.

## Theorem 1

Take $\mathrm{O} \leq \mathrm{x}, \mathrm{y} \leq 10 \leq x, y \leq 1$ in $F$ with $\mathrm{x} \neq \mathrm{O} x \neq 0$ or $\mathrm{y} \neq \mathrm{O} y \neq 0$ such that $\mathrm{x}-1>\mathrm{O} x-1>0$ if $\mathrm{x} \neq \mathrm{O} x \neq 0$ and $\mathrm{y}-1>0 \mathrm{y}-1>0$ if $\mathrm{y} \neq \mathrm{O} y \neq 0$, define the positive cone $\mathrm{P}_{\mathrm{x}, \mathrm{y}} P_{x, y}$ of $\mathrm{CF}=\mathrm{F}+\mathrm{Fi} C F=F+F i$ as follows.
$\mathrm{P}_{\mathrm{x}, \mathrm{y}}=\{\mathrm{a}+\mathrm{bi} \mid \mathrm{a} \in \mathrm{F}+,-\mathrm{xa} \leq \mathrm{nb} \leq \mathrm{ya}$ in F for all positive integers
$n\} . P_{x, y}=\{a+b i / a \in F+,-x a \leq n b \leq y a$ in $F$ for all positive integers $n\}$.
Then $\mathrm{P}_{\mathrm{x}, \mathrm{y}} P_{x, y}$ is a directed partial order on $\mathrm{CF} C F$ such that $\left(\mathrm{C}_{\mathrm{F}}, \mathrm{P}_{\mathrm{x}, \mathrm{y}}\right)\left(C F, P_{x, y}\right)$ is a partially ordered algebra over $F$ and $\mathrm{P}_{\mathrm{x}, \mathrm{y}} \cap \mathrm{F}=\mathrm{F}+P_{x}, y \cap F=F+$.

Proof
It is clear that $\mathrm{P}_{\mathrm{x}, \mathrm{y}} \cap-\mathrm{P}_{\mathrm{x}, \mathrm{y}}=\{0\} P_{x, y} \cap-P_{x, y}=\{0\}, \mathrm{P}_{\mathrm{x}, \mathrm{y}}+\mathrm{P}_{\mathrm{x}, \mathrm{y}} \subseteq \mathrm{P}_{\mathrm{x}, \mathrm{y}} P_{x, y}+P_{x, y} \subseteq P_{x, y}$ and $\mathrm{F}+\mathrm{P}_{\mathrm{x}, \mathrm{y}} \subseteq \mathrm{P}_{\mathrm{x}, \mathrm{y}} F+P_{x, y \subseteq P_{x, y} . \text { Suppose that } \mathrm{a}+\mathrm{bi}, \mathrm{c}+\mathrm{di}^{\prime} \in \mathrm{P}_{\mathrm{x}, \mathrm{y}} a+b i, c+d i \in P_{x, y} .}$ Then $\mathrm{a}, \mathrm{c} \in \mathrm{F}+a, c \in F+$, and for all positive integers $n$,
$-\mathrm{a} \leq-\mathrm{xa} \leq \mathrm{nb} \leq \mathrm{ya} \leq \mathrm{a},-\mathrm{c} \leq-\mathrm{xc} \leq \mathrm{nd} \leq \mathrm{yc} \leq \mathrm{c} .-a \leq-x a \leq n b \leq y a \leq a$,
$-c \leq-x c \leq n d \leq y c \leq c$.
We show that
$(\mathrm{a}+\mathrm{bi})(\mathrm{c}+\mathrm{di})=(\mathrm{ac}-\mathrm{bd})+(\mathrm{ad}+\mathrm{bc}) \mathrm{i} \in \mathrm{P}_{\mathrm{x}, \mathrm{y}}(\mathrm{a}+b i)(c+d i)=(a c-b d)+(a d+b c) i \in P_{x, y}$

From $-\mathrm{a} \leq \mathrm{b} \leq \mathrm{a}-\mathrm{a} \leq b \leq a,-\mathrm{c} \leq \mathrm{d} \leq \mathrm{c}-c \leq d \leq c$ and Lemma 1, we have $\mathrm{bd} \leq \mathrm{ac} b d \leq a c$, that is, ac-bd $\in \mathrm{F}+a c-b d \in F+$.

For any positive integer $n, 3 n d \leq y c, 3 n b \leq y a 3 n d \leq y c, 3 n b \leq y a$ and $\mathrm{a}, \mathrm{c} \in \mathrm{F}+a, c \in F+$ implies that $3 n a d \leq y(a c), 3 n b c \leq y(a c) 3 n a d \leq y(a c), 3 n b c \leq y(a c)$. Since $-\mathrm{a} \leq 3 \mathrm{~b} \leq \mathrm{a}-a \leq 3 b \leq a$ and $-\mathrm{c} \leq \mathrm{d} \leq \mathrm{c}-c \leq d \leq c$, Lemma 1 implies that $3 \mathrm{bd} \leq \mathrm{ac} 3 b d \leq a c$, so $3 \mathrm{ybd} \leq \mathrm{y}(\mathrm{ac}) 3 y b d \leq y(a c)$. Hence $3 \mathrm{nad}+3 \mathrm{nbc}+3 \mathrm{ybd} \leq \mathrm{y}(\mathrm{ac})+\mathrm{y}(\mathrm{ac})+\mathrm{y}(\mathrm{ac})=3 \mathrm{y}(\mathrm{ac}) .3 n a d+3 n b c+3 y b d \leq y(a c)+y(a$ c) $+y(a c)=3 y(a c)$.

Since $3 \in \mathrm{~K}+3 \in K+$ and $K$ is totally ordered, ${ }_{13} \in \mathrm{~K}+\subseteq \mathrm{F}+13 \in K+\subseteq F+$, so $3(\mathrm{nad}+\mathrm{nbc}+\mathrm{ybd}) \leq 3 y(\mathrm{ac}) 3(n a d+n b c+y b d) \leq 3 y(a c)$ implies that nad+nbc+ybd $\leq \mathrm{y}(\mathrm{ac}) n a d+n b c+y b d \leq y(a c)$. Therefore $\mathrm{n}(\mathrm{ad}+\mathrm{bc}) \leq \mathrm{y}(\mathrm{ac}-\mathrm{bd}) n(a d+b c) \leq y(a c-b d)$ for all positive integers $n$. Similarly $-\mathrm{x}(\mathrm{ac}-\mathrm{bd}) \leq \mathrm{n}(\mathrm{ad}+\mathrm{bc})-x(a c-b d) \leq n(a d+b c)$. We have proved that (a+bi)(c+di) $\in \mathrm{P}_{\mathrm{x}, \mathrm{y}}(a+b i)(c+d i) \in P_{x, y}$. Thus $\mathrm{P}_{\mathrm{x}, \mathrm{y}} P_{x, y}$ is a partial order on $\mathrm{CF} C F$ with respect to which $\mathrm{CF} C F$ is a partially ordered algebra. Clearly $\mathrm{P}_{\mathrm{x}, \mathrm{y}} \cap \mathrm{F}=\mathrm{F}+P_{x}, y \cap F=F+$.

We verify that $\mathrm{P}_{\mathrm{x}, \mathrm{y}} P_{x, y}$ is a directed partial order on CF$C F$. Suppose that $\mathrm{y} \neq 0 \mathrm{O} \neq 0$. A similar argument could be used in the case $\mathrm{x} \neq \mathrm{O} x \neq 0$. Let a+bi $\in \mathrm{CFa}+b i \in C F$. Since $F$ is directed, $a$ is a difference of two elements in $\mathrm{F}+F+$, so $a$ is a difference of two elements in $\mathrm{P}_{\mathrm{x}, \mathrm{y}} P_{x, y}$ since $\mathrm{F}+\subseteq \mathrm{P}_{\mathrm{x}, \mathrm{y}} F+\subseteq P_{x, y}$. Consider $\mathrm{bi}=\mathrm{b}_{1} \mathrm{i}-\mathrm{b}_{2} \mathrm{i} b i=b 1 i-b 2 i$, where $\mathrm{b}_{1}, \mathrm{~b}_{2}>0 b 1, b 2>0$ in $F$. Since $K$ is a non-archimedean totally ordered field, there exists $\mathrm{z} \in \mathrm{K}+z \in K+$ such that $\mathrm{n} 1 \leq \mathrm{zn} 1 \leq z$ for all positive integers $n$. Let $\mathrm{w}=\mathrm{y}-1 \mathrm{~b}_{1} w=y-1 b 1$. Then $\mathrm{w}, \mathrm{wz} \in \mathrm{F}+w, w z \in F+$ and for all positive integers $n$, $-\mathrm{x}(\mathrm{wz}) \leq 0 \leq \mathrm{nb}_{1} \leq \mathrm{b}_{1 \mathrm{Z}}=\mathrm{y}(\mathrm{wz})-x(w z) \leq 0 \leq n b 1 \leq b 1 z=y(w z)$, that is, $\mathrm{wz}+\mathrm{b}_{1} \mathrm{i} \in \mathrm{P}_{\mathrm{x}, \mathrm{y}} w z+b 1 i \in P_{x, y}$. Thus bii=(wz+bii) $-\mathrm{wzb1} 1=(w z+b 1 i)-w z$ is a difference of two positive elements in CFCF. Similarly b2ib2i is a difference of two positive elements. Hence bi is a difference of two positive elements in

CF $C F$, so $\mathrm{a}+\mathrm{bi} \mathrm{a}+b i$ is a difference of two positive elements. Therefore $\mathrm{P}_{\mathrm{x}, \mathrm{y}} P_{x, y}$ is directed.

## Remark 1

If the partial order on $F$ is a lattice order, then $0<x<10<x<1$ implies that $\mathrm{x}-1>0 x-1>0$ since $\mathrm{x}<1 x<1$ implies that $x$ is an $f$-element [3, Theroem 1.20(2)].

Next we consider quaternions over $F$. Recall that $\mathrm{HF}=\mathrm{F}+\mathrm{Fi}+\mathrm{Fj}+\mathrm{Fk} H F=F+F i+F j+F k$ as a vector space over $F$ with the multiplication as follows.
$=\left(a_{0}+a_{1} i+a_{2} j+a_{3} k\right)\left(b_{0}+b_{1} i+b_{2} j+b_{3} k\right)\left(a_{0} b_{0}-a_{1} b_{1}-a_{2} b_{2}-a_{3} b_{3}\right)+\left(a_{0} b_{1}+a_{1} b_{o}+a\right.$ $\left.{ }_{2} b_{3}-a_{3} b_{2}\right) i+\left(a_{0} b_{2}+a_{2} b_{0}+a_{3} b_{1}-a_{1} b_{3}\right) j+\left(a_{0} b_{3}+a_{3} b_{o}+a_{1} b_{2}-a_{2} b_{1}\right) k,\left(a 0+a 1 i+a_{2} j\right.$ $+a 3 k)(b 0+b 1 i+b 2 j+b 3 k)=(a 0 b 0-a 1 b 1-a 2 b 2-a 3 b 3)+(a 0 b 1+a 1 b 0+a 2 b 3-a 3$ $b 2) i+(a 0 b 2+a 2 b 0+a 3 b 1-a 1 b 3) j+(a 0 b 3+a 3 b 0+a 1 b 2-a 2 b 1) k$,
where ai,bi $\in$ Fai,bi $\in F$.

## Theorem 2

Take $\mathrm{O}<\mathrm{x} \leq 10<x \leq 1$ in $F$ such that $\mathrm{x}-1>0 \mathrm{x}-1>0$, define the positive cone $\mathrm{P}_{\mathrm{x}} P_{x}$ of $\mathrm{HF} H F$ as follows.
$P_{x}=\left\{a_{o}+a_{11}+a_{2 j}+a_{3} k \mid a_{o} \in F+\right.$ and $-x a_{0} \leq n_{1} \leq x a_{0},-x a_{0} \leq n_{2} \leq x a o$, $-x a_{0} \leq$ na $_{3} \leq x a o$ in F for all positive integers n$\} . P_{x}=\{a 0+a 1 i+a 2 j+a 3 \mathrm{k} /$ $a 0 \in F+$ and $-x a 0 \leq n a 1 \leq x a 0,-x a 0 \leq n a 2 \leq x a 0,-x a 0 \leq n a 3 \leq x a 0$ in $F$ for all positive integers n\}.

Then $\mathrm{P}_{x} P_{x}$ is a directed partial order on $\mathrm{HF} H F$ such that $\left(\mathrm{HF}, \mathrm{P}_{x}\right)\left(H F, P_{x}\right)$ is a partially ordered algebra over $F$ and $\mathrm{P} \times \mathrm{F}=\mathrm{F}+P \times \cap F=F+$.

## Proof

It is clear that $P_{x} \cap-P_{x}=\{0\} P_{x} \cap-P_{x}=\{0\}, P_{x}+P_{x} \subseteq P_{x} P_{x}+P_{x} \subseteq P_{x}$ and $\mathrm{F}_{+} \mathrm{P}_{\mathrm{x}} \subseteq \mathrm{P}_{\mathrm{x}} F+P_{x} \subseteq P_{x}$. We show that $\mathrm{P}_{\mathrm{x}} \mathrm{P}_{\mathrm{x}} \subseteq \mathrm{P}_{\mathrm{x}} P_{x} P_{x} \subseteq P_{x}$. Suppose that $\mathrm{a}_{\mathrm{o}}+\mathrm{a}_{1} 1+\mathrm{a}_{2} j+\mathrm{a}_{3} \mathrm{k}, \mathrm{b}_{0}+\mathrm{b}_{1} i+\mathrm{b}_{2} \mathrm{j}+\mathrm{b}_{3} \mathrm{k} \in \mathrm{P}_{\mathrm{x}} \mathrm{a}_{0}+\mathrm{a}_{1} 1 i+\mathrm{a} 2 j+\mathrm{a} 3 k, b 0+b 1 i+b 2 j+b 3 k \in P_{x}$. We check that
$\left(\mathrm{a}_{0}+\mathrm{a}_{1} \mathrm{i}+\mathrm{a}_{2} \mathrm{j}+\mathrm{a}_{3} \mathrm{k}\right)\left(\mathrm{bo}+\mathrm{b}_{1} \mathrm{i}+\mathrm{b}_{2} \mathrm{j}+\mathrm{b}_{3} \mathrm{k}\right) \in \mathrm{P}_{\mathrm{x}}(\mathrm{a} 0+\mathrm{a} 1 i+\mathrm{a} 2 j+\mathrm{a} 3 \mathrm{k})(b 0+b 1 i+b 2 j+b 3 k)$ $\in P_{x}$. Since
 we have for all positive integers $n$,

$-x a 0 \leq n a 2 \leq x a 0,-x a 0 \leq n a 3 \leq x a 0$,
and

$-x b 0 \leq n b 1 \leq x b 0,-x b 0 \leq n b 2 \leq x b 0,-x b 0 \leq n b 3 \leq x b 0$.
We first check that $a_{o b o}-a_{1} b_{1}-a_{2} b_{2}-a_{3} b_{3} \geq 0 a_{0} 0-a 1 b 1-a 2 b 2-a 3 b 3 \geq 0$ in $F$.
From $-\mathrm{a}_{0} \leq-x \mathrm{a}_{0} \leq 3 \mathrm{a}_{1} \leq x \mathrm{a}_{0} \leq \mathrm{a}_{0}-a 0 \leq-x a 0 \leq 3 a 1 \leq x a 0 \leq a 0$,
$-\mathrm{b}_{0} \leq-\mathrm{xb}_{0} \leq \mathrm{b}_{1} \leq \mathrm{xb}_{0} \leq \mathrm{b}_{\circ}-b 0 \leq-x b 0 \leq b 1 \leq x b 0 \leq b 0$ and Lemma 1, we have $a_{o b} \geq 3 a_{1} b_{1} a 0 b 0 \geq 3 a 1 b 1$, that is, $a_{o} b_{o}-3 a_{1} b_{1} \geq 0 a 0 b 0-3 a 1 b 1 \geq 0$. Similarly, aobo-3a2b $2 \geq 0 a 0 b 0-3 a 2 b 2 \geq 0$ and aobo $-3 \mathrm{a}_{3} \mathrm{~b}_{3} \geq 0 a 0 b 0-3 a 3 b 3 \geq 0$. Adding those three inequalities together, we obtain $3 a_{o} b_{0}-3 a_{1} b_{1}-3 a_{2} b_{2}-3 a_{3} b_{3} \geq 03 a 0 b 0-3 a_{1} b 1-3 a 2 b 2-3 a 3 b 3 \geq 0$. Then multiplying the both sides by 1313 we get $\mathrm{aobo}_{0}-\mathrm{a}_{1} \mathrm{~b}_{1}-\mathrm{a}_{2} \mathrm{~b}_{2}-\mathrm{a}_{3} \mathrm{~b}_{3} \geq 0 a 0 b 0-a 1 b 1-a 2 b 2-a 3 b 3 \geq 0$.

For simplicity, let
$\mathrm{w}=\mathrm{x}\left(\mathrm{aobo}_{0}-\mathrm{a}_{1} \mathrm{~b}_{1}-\mathrm{a}_{2} \mathrm{~b}_{2}-\mathrm{a}_{3} \mathrm{~b}_{3}\right) w=x\left(a_{0} 0 b 0-a_{1} b_{1}-a_{2} b 2-a 3 b 3\right)$. We next show that for all positive integers $n$,
$-\mathrm{w} \leq \mathrm{n}\left(\mathrm{aob}_{1}+\mathrm{a}_{1} \mathrm{~b}_{o}+\mathrm{a}_{2} \mathrm{~b}_{3}-\mathrm{a}_{3} \mathrm{~b}_{2}\right) \leq \mathrm{w} .-w \leq n(a 0 b 1+a 1 b 0+a 2 b 3-a 3 b 2) \leq w$.
Consider $\mathrm{n}\left(\mathrm{aob}_{1}+\mathrm{a}_{1} \mathrm{~b}_{0}+\mathrm{a}_{2} \mathrm{~b}_{3}-\mathrm{a}_{3} \mathrm{~b}_{2}\right) \leq \mathrm{w} n(a 0 b 1+a 1 b 0+a 2 b 3-a 3 b 2) \leq w$ first. We divide the calculations into several steps. Let $n$ be a positive integer.

- 1. 
- Since $7 \mathrm{nb} 1 \leq x b o 7 n b 1 \leq x b 0$ and $\mathrm{ao} \geq 0 a 0 \geq 0$, we have $7 n a o b_{1} \leq x a_{o}$ b $^{2 n a 0 b 1 \leq x a 0 b 0 . ~}$
- 2. 
- Since 7 na1 $\leq x a o 7 n a 1 \leq x a 0$ and $\mathrm{bo} \geq 0 b 0 \geq 0$, we have $7 n a 1 b o \leq x a o b o 7 n a 1 b 0 \leq x a 0 b 0$.
- 3 .
- Since $-\mathrm{a}_{0} \leq 7 \mathrm{na}_{2} \leq \mathrm{a}_{0}-a 0 \leq 7 n a 2 \leq a 0$ and $-x \mathrm{~b}_{0} \leq \mathrm{b}_{3} \leq x \mathrm{~b}_{0}-x b 0 \leq b 3 \leq x b 0$, Lemma 1 implies that 7 na2b $_{3} \leq x a o b o 7 n a 2 b 3 \leq x a 0 b 0$.
- Since $-\mathrm{a}_{0} \leq 7 \mathrm{na}_{3} \leq \mathrm{a}_{0}-a 0 \leq 7 n a 3 \leq a 0$ and $-x \mathrm{~b}_{0} \leq \mathrm{b}_{2} \leq x \mathrm{xb}_{0}-x b 0 \leq b 2 \leq x b 0$, Lemma 1 implies that $7 n_{3} b_{2} \geq-x a o b o 7 n a 3 b 2 \geq-x a 0 b 0$, so $-7 n a_{3} b_{2} \leq x a o b o-7 n a 3 b 2 \leq x a 0 b 0$.
- 5. 
- Since $-\mathrm{a}_{0} \leq 7 \mathrm{a}_{1} \leq \mathrm{a}_{0}-a 0 \leq 7 a 1 \leq a 0$ and $-\mathrm{b}_{0} \leq \mathrm{b}_{1} \leq \mathrm{b}_{0}-b 0 \leq b 1 \leq b 0$, Lemma 1 implies that 7a1b1 $\leq a_{0} b_{0} 7 a 1 b 1 \leq a 0 b 0$, so $7 x a 1 b_{1} \leq x a o b o 7 x a 1 b 1 \leq x a 0 b 0$. Similarly, 7xa2b $2 \leq x a o b o 7 x a 2 b 2 \leq x a 0 b 0$ and $7 x_{3} \mathrm{~b}_{3} \leq x a o b o 7 x a 3 b 3 \leq x a 0 b 0$.

Adding inequalities in the above (1) to (5) together, we have $7 n a_{o b}+7 n a_{1} b_{0}+7 n_{2} b_{3}-7 n a_{3} b_{2}+7 x a_{1} b_{1}+7$ xa $_{2} b_{2}+7 x_{3} b_{3} \leq 7 x a_{0} b_{0}, 7 n a 0 b 1+7$ na1b0+7na2b3-7na3b2+7xa1b1+7xa2b2+7xa3b3 $57 x a 0 b 0$,
and hence, by multiplying both sides of this inequality by 1717 ,
naob ${ }_{1}+n a_{1} b_{0}+$ na $_{2} b_{3}-n_{3} b_{2}+x_{1} b_{1}+x_{2} b_{2}+x_{3} b_{3} \leq x a o b o . n a 0 b 1+n a 1 b 0+n a 2$ b3-na3b2+xa1b1+xa2b2+xа3b3 $3 \leq x a 0 b 0$.

It follows that
$\mathrm{n}\left(\mathrm{aob}_{1}+\mathrm{a}_{1} \mathrm{~b}_{0}+\mathrm{a}_{2} \mathrm{~b}_{3}-\mathrm{a}_{3} \mathrm{~b}_{2}\right) \leq \mathrm{x}\left(\mathrm{a}_{3} \mathrm{~b}_{0}-\mathrm{a}_{1} \mathrm{~b}_{1}-\mathrm{a}_{2} \mathrm{~b}_{2}-\mathrm{a}_{3} \mathrm{~b}_{3}\right) n\left(\mathrm{a}_{0} 0 b_{1}+\mathrm{a}_{1} \mathrm{~b}_{0}+\mathrm{a}_{2} \mathrm{~b}_{3}-\mathrm{a} 3\right.$ b2) $\leq x(a 0 b 0-a 1 b 1-a 2 b 2-a 3 b 3)$.

Similarly we prove
$-\mathrm{w} \leq \mathrm{n}\left(\mathrm{aob}_{1}+\mathrm{a}_{1} \mathrm{~b}_{0}+\mathrm{a}_{2} \mathrm{~b}_{3}-\mathrm{a}_{3} \mathrm{~b}_{2}\right)-w \leq n\left(a_{0} b_{1}+a_{1 b} 0+a_{2} b_{3}-a 3 b 2\right)$. Let $n$ be a positive integer.

- 1. 
- Since $-x b_{0} \leq 7 n b_{1}-x b 0 \leq 7 n b 1$ and $\mathrm{a}_{0} \geq 0 a 0 \geq 0$, we have that $-x a o b o \leq 7 n a o b_{1}-x a 0 b 0 \leq 7 n a 0 b 1$.
- 2. 
- Since $-x a 0 \leq 7 n a 1-x a 0 \leq 7 n a 1$ and $\mathrm{b}_{\mathrm{o}} \geq 0 b 0 \geq 0$, we have that $-x a o b o \leq 7 n a 1 b o-x a 0 b 0 \leq 7 n a 1 b 0$.
- 3. 
- Since $-\mathrm{a}_{0} \leq 7 \mathrm{na}_{2} \leq \mathrm{a}_{0}-a 0 \leq 7 n a 2 \leq a 0$ and $-\mathrm{xb} \mathrm{o}_{0} \leq \mathrm{b}_{3} \leq x \mathrm{xb}_{0}-x b 0 \leq b 3 \leq x b 0$, Lemma 1 implies that $-x a o b o \leq 7 n a 2 b_{3}-x a 0 b 0 \leq 7 n a 2 b 3$.
- Since $-\mathrm{a}_{0} \leq 7 \mathrm{na}_{3} \leq \mathrm{a}_{0}-a 0 \leq 7 n a 3 \leq a 0$ and $-x \mathrm{~b}_{0} \leq \mathrm{b}_{2} \leq x \mathrm{bb}_{0}-x b 0 \leq b 2 \leq x b 0$, Lemma 1 implies that $7 \mathrm{na}_{3} \mathrm{~b}_{2} \leq x \mathrm{xa}_{0} \mathrm{~b}_{0} 7 n a 3 b 2 \leq x a 0 b 0$, so $-x a o b o \leq-7$ na $_{3} b_{2}-x a 0 b 0 \leq-7 n a 3 b 2$.
- 5. 
- Since $-\mathrm{a}_{0} \leq 7 \mathrm{a}_{1} \leq \mathrm{a}_{0}-a 0 \leq 7 a 1 \leq a 0$ and $-\mathrm{b}_{0} \leq \mathrm{b}_{1} \leq \mathrm{b}_{0}-b 0 \leq b 1 \leq b 0$, we have $-\mathrm{a}_{0} \leq 7 \mathrm{a}_{1} \leq \mathrm{a}_{0}-a 0 \leq 7 a 1 \leq a 0$ and $-\mathrm{b}_{0} \leq-\mathrm{b}_{1} \leq \mathrm{b}_{0}-b 0 \leq-b 1 \leq b 0$, so Lemma 1 implies that -aobo $\leq-7 a_{1} b_{1}-a 0 b 0 \leq-7 a 1 b 1$, so $-x a o b o \leq-7 x_{1} b_{1}-x a 0 b 0 \leq-7 x a 1 b 1$. Similarly, $-x a o b o \leq-7 x a_{2} b_{2}-x a 0 b 0 \leq-7 x a 2 b 2$ and $-x a o b o \leq-7$ xa $_{3} \mathrm{~b}_{3}-x a 0 b 0 \leq-7 x a 3 b 3$.

Adding up the above inequalities, we have
$-7 x a_{o} b_{0} \leq 7 n a_{o} b_{1}+7 n_{1} b_{o}+7 n a z_{2} b_{3}-7 n_{3} b_{2}-7 x a_{1} b_{1}-7 x_{2} b_{2}-7 x a_{3} b_{3},-7 x a 0 b 0$ $\leq 7 n a 0 b 1+7 n a 1 b 0+7 n a 2 b 3-7 n a 3 b 2-7 x a 1 b 1-7 x a 2 b 2-7 x a 3 b 3$,
and hence
$-x a_{o} b_{0} \leq n a o b_{1}+n a 1 b_{0}+n a_{2} b_{3}-n_{3} b_{2}-x_{1} b_{1}-7 x a_{2} b_{2}-x_{3} b_{3},-x a 0 b 0 \leq n a 0 b 1+$ na1b0+na2b3-na3b2-xa1b1-7xa2b2-xa3b3,
that is,
$-x\left(\mathrm{aob}_{0}-\mathrm{a}_{1} \mathrm{~b}_{1}-\mathrm{a}_{2} \mathrm{~b}_{2}-\mathrm{a}_{3} \mathrm{~b}_{3}\right) \leq \mathrm{n}\left(\mathrm{aob}_{1}+\mathrm{a}_{1} \mathrm{~b}_{0}+\mathrm{a}_{2} \mathrm{~b}_{3}-\mathrm{a}_{3} \mathrm{~b}_{2}\right)-x\left(a_{0} 0 b_{0}-\mathrm{a}_{1} \mathrm{~b}_{1}-\mathrm{a}_{2} \mathrm{bb}_{2}-\right.$ $a 3 b 3) \leq n(a 0 b 1+a 1 b 0+a 2 b 3-a 3 b 2)$.

By similar calculations, we have
$-\mathrm{w} \leq \mathrm{n}\left(\mathrm{aob}_{2}+\mathrm{a}_{2} \mathrm{~b}_{0}+\mathrm{a}_{3} \mathrm{~b}_{1}-\mathrm{a}_{1} \mathrm{~b}_{3}\right) \leq \mathrm{w},-w \leq n(a 0 b 2+a 2 b 0+a 3 b 1-a 1 b 3) \leq w$,
and
$-\mathrm{w} \leq \mathrm{n}\left(\mathrm{aob}_{3}+\mathrm{a}_{3} \mathrm{~b}_{0}+\mathrm{a}_{1} \mathrm{~b}_{2}-\mathrm{a}_{2} \mathrm{~b}_{1}\right) \leq \mathrm{w} .-w \leq n\left(a 0 b 3+a 3 b 0+a_{1} b 2-a 2 b 1\right) \leq w$.
Therefore $\mathrm{P}_{\mathrm{x}} \mathrm{P}_{\mathrm{x}} \subseteq \mathrm{P}_{\mathrm{x}} P_{x} P_{x} \subseteq P_{x}$, so $\left(\mathrm{HF}, \mathrm{P}_{\mathrm{x}}\right)\left(H F, P_{x}\right)$ is a partially ordered algebra over $F$. It is straightforward to verify that $\mathrm{P}_{\mathrm{x}} \cap \mathrm{F}=\mathrm{F}+P x \cap F=F+$.

Finally we show that $\mathrm{P}_{\mathrm{x}} P_{x}$ is a directed partial order on $\mathrm{HF} H F$. Take $\mathrm{a} \in \mathrm{F} a \in F$. Since $F$ is directed, $\mathrm{a}=\mathrm{b}-\mathrm{c} a=b-c$, where $\mathrm{b}, \mathrm{c} \in \mathrm{F}+b, c \in F+$. Since $\mathrm{F}+\subseteq \mathrm{P}_{\mathrm{x}} F+\subseteq P_{X}, a$ is a difference of two positive elements in (HF, $\mathrm{P}_{\mathrm{x}}$ ) $\left(H F, P_{x}\right)$. Consider $\mathrm{ai}=\mathrm{bi}-\mathrm{ci} a i=b i-c i$. Since $K$ is a non-archimedean totally ordered field, there exists $\mathrm{z} \in \mathrm{K}+z \in K+$ such that $\mathrm{n} 1 \leq \mathrm{zn} 1 \leq z$ for all positive integers $n$. Let
$\mathrm{v}=\mathrm{x}-1 \mathrm{~b} v=x-1 b$. Then $\mathrm{v}, \mathrm{vz} \in \mathrm{F}+v, v z \in F+$ and for all positive integers $n$, $\mathrm{nb} \leq \mathrm{bz}=\mathrm{x}(\mathrm{vz}) n b \leq b z=x(v z)$, that is, $\mathrm{vz}+\mathrm{bi} \in \mathrm{P}_{\mathrm{x}} v z+b i \in P_{x}$. Thus
$\mathrm{bi}=(\mathrm{vz}+\mathrm{bi})-\mathrm{vz} b i=(\mathrm{vz}+b i)-v z$ is a difference of two positive elements in ( $\mathrm{HF}, \mathrm{P}_{\mathrm{x}}$ ) $\left(H F, P_{X}\right)$. Similarly, $c i$ is also a difference of two positive elements in $(\mathrm{HF}, \mathrm{Px})\left(H F, P_{x}\right)$, and hence $\mathrm{ai}=\mathrm{bi}-\mathrm{ci} a i=b i-c i$ is a difference of two positive elements in (HF, Px $)(H F, P x)$. The same argument may be used to show that $a j$ and $a k$ are also a difference of two positive elements in ( $\mathrm{HF}, \mathrm{P}_{\mathrm{x}}$ ) $\left(H F, P_{x}\right)$. It follows that each element in HFHF is a difference of two positive elements in $\left(\mathrm{H}_{\mathrm{F}}, \mathrm{P}_{\mathrm{x}}\right)\left(H F, P_{x}\right)$, that is, $\mathrm{P}_{\mathrm{x}} P_{x}$ is directed. This completes the proof of Theorem 2. $\square \square$

In [1], Birkhoff asked if $\mathrm{H}=\mathrm{Hr} H=H R$ can be made into a directed algebra over $\mathrm{R} R$ with the usual total order, where $\mathrm{R} R$ is the field of real numbers. In the following, we show that the answer is no. More generally, for any totally ordered subfield $T$ of $\mathrm{R} R$, Нт $H T$ cannot be a directed algebra over $T$.

## Theorem 3

$\mathrm{H} H$ cannot be a directed algebra over $\mathrm{R} R$ with the usual total order.

## Proof

We will suppose that $\mathrm{H} H$ is a directed algebra over $\mathrm{R} R$ and derive a contradiction.

We first show that if $\mathrm{w}=\mathrm{a}_{0}+\mathrm{a}_{1} \mathrm{i}+\mathrm{a}_{2} \mathrm{j}+\mathrm{a}_{3} \mathrm{k}>0 \mathrm{w}=\mathrm{a} 0+\mathrm{a}_{1} 1 i+a_{2 j}+\mathrm{a}_{3} k>0$ in $\mathrm{H} H$, then $\mathrm{ao}>\mathrm{Oa} 0>0$ in $\mathrm{R} R$. Note that
$\mathrm{w}_{2}-2 \mathrm{a}_{0}=-\left(\mathrm{a}_{20}+\mathrm{a}_{21}+\mathrm{a}_{22}+\mathrm{a}_{23}\right) w 2-2 \mathrm{a} 0 w=-\left(\mathrm{a} 02+\mathrm{a}_{12}+\mathrm{a}_{2} 2+\mathrm{a} 32\right)$ by direct calculation, and that if $\mathrm{O}<\mathrm{a} 0<a$ in $\mathrm{R} R$ and $\mathrm{O}<\mathrm{z} O<z$ in $\mathrm{H} H$, then $\mathrm{O}<\mathrm{az} 0<a z$ in $\mathrm{H} H$ because $\mathrm{H} H$ is a partially ordered algebra over $\mathrm{R} R$ that has no divisor of zero.

Now suppose by the way of contradiction that ao $\leq 0 a 0 \leq 0$ in $R R$. Then $-\mathrm{a} 0 \geq 0-a 0 \geq 0$ in $\mathrm{R} R$, and thus since $\mathrm{w}>\mathrm{O} w>0$ in $\mathrm{H} H$, $\mathrm{w}_{2}-2 \mathrm{a} o w \geq 0 \mathrm{w} 2-2 a 0 w \geq 0$ in $\mathrm{H} H$. That is, $-\left(\mathrm{a}_{20}+\mathrm{a}_{21}+\mathrm{a}_{22}+\mathrm{a}_{23}\right) \geq 0-\left(a 02+a_{12}+a_{22}+a_{32}\right) \geq 0$ in H $H$, and hence $-\left(\mathrm{a}_{20}+\mathrm{a}_{21}+\mathrm{a}_{22}+\mathrm{a}_{23}\right) \mathrm{w}>\mathrm{O}-(\mathrm{a} 02+\mathrm{a} 12+\mathrm{a} 22+\mathrm{a} 32) w>0$ in $\mathrm{H} H$. But since $\left(\mathrm{a}_{20}+\mathrm{a}_{21}+\mathrm{a}_{22}+\mathrm{a}_{23}\right) \geq \mathrm{O}\left(a 02+\mathrm{a}_{12}+\mathrm{a}_{22}+\mathrm{a}_{32}\right) \geq 0$ in $\mathrm{R} R$, $\left(\mathrm{a}_{20}+\mathrm{a}_{21}+\mathrm{a}_{22}+\mathrm{a}_{23}\right) \mathrm{w} \geq \mathrm{O}\left(\mathrm{a}_{02}+\mathrm{a} 12+\mathrm{a} 22+\mathrm{a} 32\right) w \geq 0$ in $\mathrm{H} H$ as well, and therefore $\left(\mathrm{a}_{20}+\mathrm{a}_{21}+\mathrm{a}_{22}+\mathrm{a}_{23}\right) \mathrm{w}=\mathrm{O}\left(a 02+\mathrm{a}_{12}+\mathrm{a}_{22}+\mathrm{a} 32\right) w=0$. So since
$\mathrm{w} \neq \mathrm{O} w \neq 0$ and $\mathrm{H} H$ is a division ring, we must have $\mathrm{a}_{20}+\mathrm{a}_{21}+\mathrm{a}_{22}+\mathrm{a}_{23}=\mathrm{O} a_{02}+\mathrm{a}_{12}+\mathrm{a}_{22}+\mathrm{a} 32=0$. But then $\mathrm{a}_{0}=\mathrm{a}_{1}=\mathrm{a}_{2}=\mathrm{a}_{3}=0 \mathrm{a} 0=a 1=a 2=a 3=0$ because $\mathrm{a}_{0}, \mathrm{a}_{1}, \mathrm{a}_{2} a 0, a 1, a 2$, and $\mathrm{a}_{3} a 3$ are all in $\mathrm{R} R$, and hence $\mathrm{w}=\mathrm{a} 0+\mathrm{a}_{1} \mathrm{i}+\mathrm{a}_{2} \mathrm{j}+\mathrm{a}_{3} \mathrm{k}=\mathrm{O} w=a 0+\mathrm{a}_{1} 1 i+\mathrm{a}_{2} j+\mathrm{a} 3 \mathrm{k}=0$, a contradiction. It follows that $\mathrm{ao}>\mathrm{O} 0>0$ in $\mathrm{R} R$.

Since the partial order on $\mathrm{H} H$ is directed, there exists $\mathrm{z}=\mathrm{a}+\mathrm{bi}+\mathrm{cj}+\mathrm{dk}>\mathrm{O} z=a+b i+c j+d k>0$ in $\mathrm{H} H$ with $\mathrm{z} \notin \mathrm{R} z \notin R$. For instance, $\mathrm{i}=\mathrm{Z}_{1}-\mathrm{Z}_{2} i=z 1-z 2$, where $\mathrm{Z} 1, \mathrm{Z}_{2} z 1, z 2$ are positive in $\mathrm{H} H$. Clearly $\mathrm{Z}_{1}, \mathrm{Z}_{2} z 1, z 2$ cannot be both in $\mathrm{R} R$. The argument above shows that $\mathrm{a}>0 \mathrm{a}>0$ in $\mathrm{R} R$. Then since $\mathrm{R} R$ is totally ordered, $\mathrm{a}-1>0 a-1>0$ in $\mathrm{R} R$ and hence $\mathrm{a}-1 \mathrm{Z}=1+(\mathrm{a}-1 \mathrm{~b}) \mathrm{i}+(\mathrm{a}-1 \mathrm{c}) \mathrm{j}+(\mathrm{a}-1 \mathrm{~d}) \mathrm{k}>\mathrm{O} a-1 z=1+(a-1 b) i+(a-1 c) j+(a-1 d) k>0$ in $\mathrm{H} H$. Suppose that $\mathrm{a}-\mathrm{b}=\mathrm{s}, \mathrm{a}-1 \mathrm{c}=\mathrm{t}, \mathrm{a}-1 \mathrm{~d}=\mathrm{u} a-1 b=s, a-1 c=t, a-1 d=u$. Then we have $\mathrm{w}=1+\mathrm{si}+\mathrm{tj}+\mathrm{uk}>\mathrm{O} w=1+s i+t j+u k>0$ in $\mathrm{H} H$ and $\mathrm{w} \notin \mathrm{R} w \notin R$. For simplicity, let $\mathrm{v}=\mathrm{si}+\mathrm{tj}+\mathrm{uk} v=s i+t j+u k$. Then $\mathrm{v}_{2}=-\left(\mathrm{s}_{2}+\mathrm{t}_{2}+\mathrm{u}_{2}\right) v 2=-(s 2+t 2+u 2)$, so $-\mathrm{v}_{2} \in \mathrm{R}_{+}-v 2 \in R+$. Therefore $-\mathrm{v}_{2} \mathrm{~W} \geq 0-v 2 w \geq 0$ in $\mathrm{H} H$, and hence $\mathrm{w}_{3}-\mathrm{v} 2 \mathrm{w}>0 \mathrm{w} 3-v 2 w>0$ in H $H$. Since
$1+3 \mathrm{v}+2 \mathrm{v}_{2}==(1+2 \mathrm{v})(1+\mathrm{v})=\left(1+2 \mathrm{v}+\mathrm{v}_{2}-\mathrm{v}_{2}\right) \mathrm{w}\left(\mathrm{w}_{2}-\mathrm{v}_{2}\right) \mathrm{w}=\mathrm{w}_{3}-\mathrm{v}_{2} \mathrm{~W}, 1+3 \mathrm{v}+2 \mathrm{v} 2=($ $1+2 v)(1+v)=(1+2 v+v 2-v 2) w=(w 2-v 2) w=w 3-v 2 w$,
we have $1+3 \mathrm{v}+2 \mathrm{v}_{2}>\mathrm{O} 1+3 \mathrm{v}+2 \mathrm{v} 2>0$ in $\mathrm{H} H$. Let $\mathrm{w}_{1}=1+3 \mathrm{v}+2 \mathrm{v}_{2} \mathrm{w} 1=1+3 \mathrm{v}+2 \mathrm{v} 2$. Then $\mathrm{w}_{1}>0 w 1>0$ in $\mathrm{H} H$, and hence
$\left(\mathrm{w}_{1}-2 \mathrm{~V}_{2}\right) \mathrm{w}=\mathrm{w}_{1} \mathrm{w}-2 \mathrm{~V}_{2} \mathrm{~W}>0(w 1-2 \mathrm{v} 2) w=w 1 w-2 v 2 w>0$ in H $H$. Since
$1+4 \mathrm{v}+3 \mathrm{v}_{2}=(1+3 \mathrm{v})(1+\mathrm{v})=\left(1+3 \mathrm{v}+2 \mathrm{v}_{2}-2 \mathrm{v}_{2}\right) \mathrm{w}=\left(\mathrm{w}_{1}-2 \mathrm{v}_{2}\right) \mathrm{w} 1+4 \mathrm{v}+3 \mathrm{v} 2=(1+3 \mathrm{v})($ $1+v)=(1+3 v+2 v 2-2 v 2) w=(w 1-2 v 2) w$
$1+4 \mathrm{v}+3 \mathrm{v}_{2}>\mathrm{O} 1+4 \mathrm{v}+3 \mathrm{v} 2>0$ in $\mathrm{H} H$. If we continue this procedure, we get that for any positive integer $n$, $(1+\mathrm{nv})(1+\mathrm{v})=1+(\mathrm{n}+1) \mathrm{v}+\mathrm{nv}_{2}>\mathrm{O}(1+n v)(1+v)=1+(n+1) v+n v 2>0$ in H $H$. Therefore since the real part of a positive element in $\mathrm{H} H$ must be positive in $\mathrm{R} R$, we must have $0 \leq 1+\mathrm{n}_{2} 0 \leq 1+n v 2$ for all positive integers $n$, so $-\mathrm{nv} 2 \leq 1-n v 2 \leq 1$ for all positive integers $n$. Then $-\mathrm{v}_{2}=0-v 2=0$, so $\mathrm{v}_{2}=0 \mathrm{v} 2=0$ since $R R$ is archimedean with respect to the total order. Hence $\mathrm{s}=\mathrm{t}=\mathrm{u}=\mathrm{O} s=t=u=0$, and $\mathrm{w} \in \mathrm{R} w \in R$, a contradiction of our observation above that $\mathrm{w} \notin \mathrm{R} w \notin R$.

Therefore $\mathrm{H} H$ cannot be a directed algebra over $\mathrm{R} R$ with the total order.

