Suppose that F is a partially ordered field with a directed partial order and K is a non-archemedean totally ordered subfield of F with $K_+=F_+\cap K$. In this note, directed partial orders are constructed for complex numbers and quaternions over F. It is also shown that real quaternions cannot be made into a directed algebra over the real field with the total order.

Let T be a non-archimedean totally ordered field and $C_T=T+Ti$ be the field of complex numbers over T, where $i^2=-1$, and $H_T=T+Ti+Tj+Tk$ be the division algebra of quaternions over T. In [4], a general method of constructing directed partial orders on C_T and H_T has been developed. The purpose of this note is to generalize the results in [4] to complex numbers and quaternions over non-archimedean partially ordered fields with a directed partial order.

We review a few definitions and the reader is referred to [2, 4] for undefined terminology and background on partially ordered rings and lattice-ordered rings (ℓ-rings). For a partially ordered ring R, the positive cone is defined as $R_+=\{r\in R\mid r\geq 0\}$. The partial order on a partially ordered ring R is called *directed* if each element of R can be written as a difference of two positive elements of R. A partially ordered ring (algebra) with a directed partial order is called a *directed ring* (*algebra*). In the following, we always assume that F is a partially ordered field with a directed partial order and K is a non-archimedean totally ordered subfield of F, so $F_+\cap K=K_+$.

Following result will be used in the proof of main results in the paper.

Lemma 1

Suppose that R is a partially ordered ring with the property that for any $r\in R$, $2r\geq 0$ implies that $r\geq 0$. Then for any $x,y\in R_+$, and $a,b\in R$, $-x\leq a\leq x$ and $-y\leq b\leq y$ implies that $-xy\leq ab\leq xy$.

Proof

From $-x\leq a\leq x$ and $-y\leq b\leq y$, we have

$$(x+a)(y-b)\geq 0\Rightarrow xy+ay-xb-ab\geq 0, \tag{1}$$

and

$$(x-a)(y+b)\geq 0 \Rightarrow xy-ay+xb-ab\geq 0. \tag{2}$$

Adding (1) and (2), we have $2(xy-ab)\geq 0$, so $xy-ab\geq 0$ and $xy\geq ab$.

From $-x\leq a\leq x$ and $-y\leq b\leq y$, we also have

$$(x+a)(y+b)\geq 0 \Rightarrow xy+ay+xb+ab\geq 0, \tag{3}$$

and

$$(x-a)(y-b)\geq 0 \Rightarrow xy-ay-xb+ab\geq 0. \tag{4}$$

Adding (3) and (4), we have $2(xy+ab)\geq 0$, so $xy+ab\geq 0$, that is, $ab\geq -xy$. □

We notice that since F contains a totally ordered subfield K, for any $a\in F$, $2a\geq 0$ implies that $a\geq 0$. In fact, since $0<2\in K$, $0<\frac{1}{2}\in K$, so $a=\frac{1}{2}(2a)\geq 0$.

Theorem 1

Take $0\leq x,y\leq 1$ in F with $x\neq 0$ or $y\neq 0$ such that $x^{-1}>0$ if $x\neq 0$ and $y^{-1}>0$ if $y\neq 0$, define the positive cone $P_{x,y}$ of $C_F=F+Fi$ as follows.

$$P_{x,y}=\{a+bi \mid a\in F^+, -xa\leq nb\leq ya \text{ in } F \text{ for all positive integers } n\}.$$

Then $P_{x,y}$ is a directed partial order on C_F such that $(C_F,P_{x,y})$ is a partially ordered algebra over F and $P_{x,y}\cap F=F^+$.

Proof

It is clear that $P_{x,y}\cap -P_{x,y}=\{0\}$, $P_{x,y}+P_{x,y}\subseteq P_{x,y}$ and $F^+P_{x,y}\subseteq P_{x,y}$. Suppose that $a+bi, c+di\in P_{x,y}$. Then $a,c\in F^+$, and for all positive integers n,

$$-a\le -xa\le nb\le ya\le a,\quad -c\le -xc\le nd\le yc\le c.$$

We show that $(a+bi)(c+di)=(ac-bd)+(ad+bc)i\in P_{x,y}$.

From $-a\le b\le a$, $-c\le d\le c$ and Lemma 1, we have $bd\le ac$, that is, $ac-bd\in F_+$.

For any positive integer n, $3nd\le yc, 3nb\le ya$ and $a,c\in F_+$ implies that $3nad\le y(ac), 3nbc\le y(ac)$. Since $-a\le 3b\le a$ and $-c\le d\le c$, Lemma 1 implies that $3bd\le ac$; so $3ybd\le y(ac)$. Hence

$$3nad+3nbc+3ybd\le y(ac)+y(ac)+y(ac)=3y(ac).$$

Since $3\in K_+$ and K is totally ordered, $\frac{1}{3}\in K_+\subseteq F_+$, so $3(nad+nbc+ybd)\le 3y(ac)$ implies that $nad+nbc+ybd\le y(ac)$. Therefore $n(ad+bc)\le y(ac-bd)$ for all positive integers n. Similarly $-x(ac-bd)\le n(ad+bc)$. We have proved that $(a+bi)(c+di)\in P_{x,y}$. Thus $P_{x,y}$ is a partial order on C_F with respect to which C_F is a partially ordered algebra. Clearly $P_{x,y}\cap F=F_+$.

We verify that $P_{x,y}$ is a directed partial order on C_F. Suppose that $y\neq 0$. A similar argument could be used in the case $x\neq 0$. Let $a+bi\in C_F$. Since F is directed, a is a difference of two elements in F_+, so a is a difference of two elements in $P_{x,y}$ since $F_+\subseteq P_{x,y}$. Consider $bi=b_1i-b_2i$, where $b_1,b_2>0$ in F. Since K is a non-archimedean totally ordered field, there exists $z\in K_+$ such that $n1\le z$ for all positive integers n. Let $w=y^{-1}b_1$. Then $w, wz\in F_+$ and for all positive integers n, $-x(wz)\le 0\le nb_1\le b_1z=y(wz)$, that is, $wz+b_1i\in P_{x,y}$. Thus $b_1i=(wz+b_1i)-wz$ is a difference of two positive elements in C_F. Similarly b_2i is a difference of two positive elements. Hence bi is a difference of two positive elements in

C_F, so $a+bi$ is a difference of two positive elements. Therefore $P_{x,y}$ is directed. □

Remark 1

If the partial order on *F* is a lattice order, then $0<x<1$ implies that $x^{-1}>0$ since $x<1$ implies that *x* is an *f*-element [3, Theroem 1.20(2)].

Next we consider quaternions over *F*. Recall that $H_F=F+Fi+Fj+Fk$ as a vector space over *F* with the multiplication as follows.

$$(a_0+a_1i+a_2j+a_3k)(b_0+b_1i+b_2j+b_3k)=(a_0b_0-a_1b_1-a_2b_2-a_3b_3)+(a_0b_1+a_1b_0+a_2b_3-a_3b_2)i+(a_0b_2+a_2b_0+a_3b_1-a_1b_3)j+(a_0b_3+a_3b_0+a_1b_2-a_2b_1)k,$$

where $a_i,b_i\in F$.

Theorem 2

Take $0<x\le 1$ in *F* such that $x^{-1}>0$, define the positive cone P_x of H_F as follows.

$$P_x=\{a_0+a_1i+a_2j+a_3k \mid a_0\in F^+ \text{ and } -xa_0\le na_1\le xa_0, -xa_0\le na_2\le xa_0, -xa_0\le na_3\le xa_0 \text{ in } F \text{ for all positive integers } n\}.$$

Then P_x is a directed partial order on H_F such that (H_F,P_x) is a partially ordered algebra over *F* and $P_x\cap F=F^+$.

Proof

It is clear that $P_x\cap -P_x=\{0\}$, $P_x+P_x\subseteq P_x$ and $F^+P_x\subseteq P_x$. We show that $P_xP_x\subseteq P_x$. Suppose that $a_0+a_1i+a_2j+a_3k, b_0+b_1i+b_2j+b_3k\in P_x$. We check that $(a_0+a_1i+a_2j+a_3k)(b_0+b_1i+b_2j+b_3k)\in P_x$. Since $a_0+a_1i+a_2j+a_3k, b_0+b_1i+b_2j+b_3k\in P_x$, we have for all positive integers *n*,

$a_0 \geq 0$, $-xa_0 \leq na_1 \leq xa_0$, $-xa_0 \leq na_2 \leq xa_0$, $-xa_0 \leq na_3 \leq xa_0$,

and

$b_0 \geq 0$, $-xb_0 \leq nb_1 \leq xb_0$, $-xb_0 \leq nb_2 \leq xb_0$, $-xb_0 \leq nb_3 \leq xb_0$.

We first check that $a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3 \geq 0$ in F.

From $-a_0 \leq -xa_0 \leq 3a_1 \leq xa_0 \leq a_0$, $-b_0 \leq -xb_0 \leq b_1 \leq xb_0 \leq b_0$ and Lemma 1, we have $a_0b_0 \geq 3a_1b_1$, that is, $a_0b_0 - 3a_1b_1 \geq 0$. Similarly, $a_0b_0 - 3a_2b_2 \geq 0$ and $a_0b_0 - 3a_3b_3 \geq 0$. Adding those three inequalities together, we obtain $3a_0b_0 - 3a_1b_1 - 3a_2b_2 - 3a_3b_3 \geq 0$. Then multiplying the both sides by $\frac{1}{3}$ we get $a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3 \geq 0$.

For simplicity, let $w = x(a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3)$. We next show that for all positive integers n,

$$-w \leq n(a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2) \leq w.$$

Consider $n(a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2) \leq w$ first. We divide the calculations into several steps. Let n be a positive integer.

1. Since $7nb_1 \leq xb_0$ and $a_0 \geq 0$, we have $7na_0b_1 \leq xa_0b_0$.
2. Since $7na_1 \leq xa_0$ and $b_0 \geq 0$, we have $7na_1b_0 \leq xa_0b_0$.
3. Since $-a_0 \leq 7na_2 \leq a_0$ and $-xb_0 \leq b_3 \leq xb_0$, Lemma 1 implies that $7na_2b_3 \leq xa_0b_0$.
4.

- Since $-a_0 \le 7na_3 \le a_0$ and $-xb_0 \le b_2 \le xb_0$, Lemma 1 implies that $7na_3b_2 \ge -xa_0b_0$, so $-7na_3b_2 \le xa_0b_0$.

- 5.

- Since $-a_0 \le 7a_1 \le a_0$ and $-b_0 \le b_1 \le b_0$, Lemma 1 implies that $7a_1b_1 \le a_0b_0$, so $7xa_1b_1 \le xa_0b_0$. Similarly, $7xa_2b_2 \le xa_0b_0$ and $7xa_3b_3 \le xa_0b_0$.

Adding inequalities in the above (1) to (5) together, we have

$$7na_0b_1+7na_1b_0+7na_2b_3-7na_3b_2+7xa_1b_1+7xa_2b_2+7xa_3b_3 \le 7xa_0b_0,$$

and hence, by multiplying both sides of this inequality by $\frac{1}{7}$,

$$na_0b_1+na_1b_0+na_2b_3-na_3b_2+xa_1b_1+xa_2b_2+xa_3b_3 \le xa_0b_0.$$

It follows that $n(a_0b_1+a_1b_0+a_2b_3-a_3b_2) \le x(a_0b_0-a_1b_1-a_2b_2-a_3b_3)$.

Similarly we prove $-w \le n(a_0b_1+a_1b_0+a_2b_3-a_3b_2)$. Let n be a positive integer.

- 1.

- Since $-xb_0 \le 7nb_1$ and $a_0 \ge 0$, we have that $-xa_0b_0 \le 7na_0b_1$.

- 2.

- Since $-xa_0 \le 7na_1$ and $b_0 \ge 0$, we have that $-xa_0b_0 \le 7na_1b_0$.

- 3.

- Since $-a_0 \le 7na_2 \le a_0$ and $-xb_0 \le b_3 \le xb_0$, Lemma 1 implies that $-xa_0b_0 \le 7na_2b_3$.

- 4.

- Since $-a_0 \le 7na_3 \le a_0$ and $-xb_0 \le b_2 \le xb_0$, Lemma 1 implies that $7na_3b_2 \le xa_0b_0$, so $-xa_0b_0 \le -7na_3b_2$.

- 5.

- Since $-a_0 \le 7a_1 \le a_0$ and $-b_0 \le b_1 \le b_0$, we have $-a_0 \le 7a_1 \le a_0$ and $-b_0 \le -b_1 \le b_0$, so Lemma 1 implies that $-a_0b_0 \le -7a_1b_1$, so $-xa_0b_0 \le -7xa_1b_1$. Similarly, $-xa_0b_0 \le -7xa_2b_2$ and $-xa_0b_0 \le -7xa_3b_3$.

Adding up the above inequalities, we have

$$-7xa_0b_0 \le 7na_0b_1 + 7na_1b_0 + 7na_2b_3 - 7na_3b_2 - 7xa_1b_1 - 7xa_2b_2 - 7xa_3b_3,$$

and hence

$$-xa_0b_0 \le na_0b_1 + na_1b_0 + na_2b_3 - na_3b_2 - xa_1b_1 - 7xa_2b_2 - xa_3b_3,$$

that is,
$-x(a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3) \le n(a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2)$.

By similar calculations, we have

$$-w \le n(a_0b_2 + a_2b_0 + a_3b_1 - a_1b_3) \le w,$$

and

$$-w \le n(a_0b_3 + a_3b_0 + a_1b_2 - a_2b_1) \le w.$$

Therefore $P_xP_x \subseteq P_x$, so (H_F, P_x) is a partially ordered algebra over *F*. It is straightforward to verify that $P_x \cap F = F_+$.

Finally we show that P_x is a directed partial order on H_F. Take $a \in F$. Since *F* is directed, $a = b - c$, where $b, c \in F_+$. Since $F_+ \subseteq P_x$, *a* is a difference of two positive elements in (H_F, P_x). Consider $a_i = b_i - c_i$. Since *K* is a non-archimedean totally ordered field, there exists $z \in K_+$ such that $n1 \le z$ for all positive integers *n*. Let

$v=x^{-1}b$. Then $v, vz \in F^+$ and for all positive integers n, $nb \le bz = x(vz)$, that is, $vz+bi \in P_x$. Thus $bi=(vz+bi)-vz$ is a difference of two positive elements in (H_F, P_x). Similarly, ci is also a difference of two positive elements in (H_F, P_x), and hence $ai=bi-ci$ is a difference of two positive elements in (H_F, P_x). The same argument may be used to show that aj and ak are also a difference of two positive elements in (H_F, P_x). It follows that each element in H_F is a difference of two positive elements in (H_F, P_x), that is, P_x is directed. This completes the proof of Theorem 2. $\square$

In [1], Birkhoff asked if $H=H_R$ can be made into a directed algebra over R with the usual total order, where R is the field of real numbers. In the following, we show that the answer is no. More generally, for any totally ordered subfield T of R, H_T cannot be a directed algebra over T.

Theorem 3

H cannot be a directed algebra over R with the usual total order.

Proof

We will suppose that H is a directed algebra over R and derive a contradiction.

We first show that if $w=a_0+a_1i+a_2j+a_3k>0$ in H, then $a_0>0$ in R. Note that $w^2-2a_0w=-(a_0^2+a_1^2+a_2^2+a_3^2)$ by direct calculation, and that if $0<a$ in R and $0<z$ in H, then $0<az$ in H because H is a partially ordered algebra over R that has no divisor of zero.

Now suppose by the way of contradiction that $a_0\le 0$ in R. Then $-a_0\ge 0$ in R, and thus since $w>0$ in H, $w^2-2a_0w\ge 0$ in H. That is, $-(a_0^2+a_1^2+a_2^2+a_3^2)\ge 0$ in H, and hence $-(a_0^2+a_1^2+a_2^2+a_3^2)w>0$ in H. But since $(a_0^2+a_1^2+a_2^2+a_3^2)\ge 0$ in R, $(a_0^2+a_1^2+a_2^2+a_3^2)w\ge 0$ in H as well, and therefore $(a_0^2+a_1^2+a_2^2+a_3^2)w=0$. So since

$w \neq 0$ and $\mathbb{H}$ is a division ring, we must have $a_0^2+a_1^2+a_2^2+a_3^2=0$. But then $a_0=a_1=a_2=a_3=0$ because a_0, a_1, a_2, and a_3 are all in $\mathbb{R}$, and hence $w=a_0+a_1i+a_2j+a_3k=0$, a contradiction. It follows that $a_0>0$ in $\mathbb{R}$.

Since the partial order on $\mathbb{H}$ is directed, there exists $z=a+bi+cj+dk>0$ in $\mathbb{H}$ with $z \notin \mathbb{R}$. For instance, $i=z_1-z_2$, where z_1, z_2 are positive in $\mathbb{H}$. Clearly z_1, z_2 cannot be both in $\mathbb{R}$. The argument above shows that $a>0$ in $\mathbb{R}$. Then since $\mathbb{R}$ is totally ordered, $a^{-1}>0$ in $\mathbb{R}$ and hence $a^{-1}z=1+(a^{-1}b)i+(a^{-1}c)j+(a^{-1}d)k>0$ in $\mathbb{H}$. Suppose that $a^{-1}b=s, a^{-1}c=t, a^{-1}d=u$. Then we have $w=1+si+tj+uk>0$ in $\mathbb{H}$ and $w \notin \mathbb{R}$. For simplicity, let $v=si+tj+uk$. Then $v^2=-(s^2+t^2+u^2)$, so $-v^2 \in \mathbb{R}^+$. Therefore $-v^2w \geq 0$ in $\mathbb{H}$, and hence $w^3-v^2w>0$ in $\mathbb{H}$. Since

$$1+3v+2v^2=(1+2v)(1+v)=(1+2v+v^2-v^2)w=(w^2-v^2)w=w^3-v^2w,$$

we have $1+3v+2v^2>0$ in $\mathbb{H}$. Let $w_1=1+3v+2v^2$. Then $w_1>0$ in $\mathbb{H}$, and hence $(w_1-2v^2)w=w_1w-2v^2w>0$ in $\mathbb{H}$. Since

$$1+4v+3v^2=(1+3v)(1+v)=(1+3v+2v^2-2v^2)w=(w_1-2v^2)w$$

$1+4v+3v^2>0$ in $\mathbb{H}$. If we continue this procedure, we get that for any positive integer n, $(1+nv)(1+v)=1+(n+1)v+nv^2>0$ in $\mathbb{H}$. Therefore since the real part of a positive element in $\mathbb{H}$ must be positive in $\mathbb{R}$, we must have $0 \leq 1+nv^2$ for all positive integers n, so $-nv^2 \leq 1$ for all positive integers n. Then $-v^2=0$, so $v^2=0$ since $\mathbb{R}$ is archimedean with respect to the total order. Hence $s=t=u=0$, and $w \in \mathbb{R}$, a contradiction of our observation above that $w \notin \mathbb{R}$.

Therefore $\mathbb{H}$ cannot be a directed algebra over $\mathbb{R}$ with the total order.